PHYS-4601 Homework 6 Due 27 Oct 2016

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Scattering and the Probability Current**

   We define the probability current $\vec{\jmath}(\vec{x},t)$ as

$$\vec{\jmath}(x,t) = \frac{i\hbar}{2m}\left(\Psi\vec{\nabla}\Psi^* - \Psi^*\vec{\nabla}\Psi\right) \tag{1}$$

   (in 1D, replace the gradient with $\partial\Psi/\partial x$).

   (a) A conserved quantity Q (this could be electric charge or total probability in quantum mechanics) with a density ρ and current $\vec{\jmath}$ satisfies the *continuity equation*

$$\frac{\partial\rho}{\partial t} = -\vec{\nabla}\cdot\vec{\jmath}\,. \tag{2}$$

   Consider an infinite square well potential ($V(x) = 0$ for $-a < x < a$, $V(x) = \infty$ for $|x| > a$). At time $t = 0$, the system is in state $|\Psi(0)\rangle = (|1\rangle + |2\rangle)/\sqrt{2}$, where $|1\rangle$ is the ground state and $|2\rangle$ is the first excited state. Find the $\Psi(x,t)$, probability density $\rho(x,t)$, and probability current $j(x,t)$, and verify that they obey the continuity equation.

   (b) To understand reflection and transmission coefficients when the potential takes different values on either side of the barrier, we should think about conservation of probability. Specifically, write the wavefunction as $Ae^{ikx} + Be^{-ikx}$ to the far left and $Ce^{ik'x}$ to the far right. Show first that the probability current on the far left splits into an incident part ($j_{inc} \propto |A|^2$) and a reflected part ($j_{ref} \propto |B|^2$) and evaluate j_{inc}, j_{ref}. Then find the transmitted probability current j_{trans}. Finally, find the reflection and transmission coefficients $R = j_{ref}/j_{inc}$, $T = j_{trans}/j_{inc}$ in terms of A, B, C, k, k'.

   (c) *from Griffiths 2.35* Consider a particle moving in a 1D potential

$$V(x) = \begin{cases} 0 & x < 0 \\ -V_0 & x \geq 0 \end{cases}\,. \tag{3}$$

   Determine the reflection and transmission coefficients for a particle incoming from the left (negative x) with energy $E = V_0/3$. Use $R = j_{ref}/j_{inc}, T = j_{trans}/j_{inc}$ and verify that $R + T = 1$.

2. **Square Well with Delta Function**

   Consider a particle in one-dimension with the potential

$$V(x) = \begin{cases} \infty & (x < -a\ ,\ x > a) \\ -\alpha\delta(x) & (-a < x < a) \end{cases}\,, \tag{4}$$

   where α is a positive constant.

   (a) This potential has one negative energy state if α is large enough. Define $E = -\hbar^2\kappa^2/2m$ and find the functional form of the wavefunction of this state (do not determine the normalization constant). Write down the transcendental equation that determines κ and discuss when it has a solution.

(b) Sketch the wavefunctions of the two stationary states with the smallest positive energies. Then sketch the same two wavefunctions in the case that α is negative. Give a mathematical justification for your answers.

3. **Sloshing in the Square Well**

   A particle moving in the 1D potential

$$V(x) = \begin{cases} \infty & \text{for } |x| > a \\ 0 & \text{for } -a < x < a \end{cases} \tag{5}$$

   is in the state $|\psi\rangle = (|1\rangle + i|2\rangle)/\sqrt{2}$ at time $t = 0$, where $|1\rangle$ is the ground state and $|2\rangle$ is first excited state.

   (a) What is the probability density to find the particle at $x = -a/2$ as a function of time?

   (b) Find the expectation value $\langle x \rangle$ as a function of time. *Hint:* angle addition formulae will be helpful.