PHYS-4601 Homework 5 Due 20 Oct 2016

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Gaussian Wavepacket Part II** *based on Griffiths 2.22*

   Here we return to the Gaussian wavepacket in 1D, here looking at the time evolution for the free particle Hamiltonian. We recall from the last assignment that the wavefunction (at some initial time) can be written as

   $$|\Psi(t=0)\rangle = \int_{-\infty}^{\infty} dx\ \left(\frac{2a}{\pi}\right)^{1/4} e^{-ax^2}|x\rangle = \int_{-\infty}^{\infty} dp\ \left(\frac{1}{2\pi a\hbar^2}\right)^{1/4} e^{-p^2/4a\hbar^2}|p\rangle\ . \tag{1}$$

   (a) Evolve this state in time. First write $\langle p|\Psi(t)\rangle$ and then show that

   $$\langle x|\Psi(t)\rangle = \frac{(2a/\pi)^{1/4}}{\sqrt{1+2i\hbar at/m}} e^{-ax^2/(1+2i\hbar at/m)}\ . \tag{2}$$

   Hint: You may want to use the trick of "completing the squares" to evaluate a Gaussian integral somewhere.

   (b) Find the probability density $|\langle x|\Psi(t)\rangle|^2$. Using the result from assignment 4 that $\langle x^2\rangle = 1/4a$ at $t=0$, find $\langle x^2\rangle$ at a later time t by inspection of the probability density. Qualitatively explain what's happening to the wavefunction as time passes.

   (c) What's the momentum-space probability density $|\langle p|\Psi(t)\rangle|^2$? Does $\langle p^2\rangle$ change in time? Does this state continue to saturate the Heisenberg uncertainty relation for $t \neq 0$?

2. **Proofs About Stationary States**

   (a) *Rephrasing Griffiths 2.1(c)* Consider the spatial part of a stationary state $\psi(x)$ (that is, $\Psi(x,t) = \psi(x)e^{-iEt/\hbar}$) and suppose that the potential is an even function of x (ie, $V(x) = V(-x)$). Show that $\psi(\vec{x})$ can be chosen to be either an even or odd function of x. *Hint:* argue that, for any $\psi(x)$ that solves the time-independent Schrödinger equation, so does $\psi(-x)$. Use that to show that the even and odd parts $\psi_\pm(x) = [\psi(x) \pm \psi(-x)]/2$ are also solutions with the same energy.

   (b) *Griffiths 2.2 rephrased* Suppose that the energy E of a stationary state in one dimension is less than the minimum value of the potential. Use the time-independent Schrödinger equation to show that the second derivative of the wavefunction always has the same sign as the wavefunction. Then use that fact to argue qualitatively that such a wavefunction cannot be normalized, proving by contradiction that E must be greater than the minimum value of the potential.

3. **Double Delta-Function Well** *based on Griffiths 2.27*

   Consider the potential

   $$V(x) = -\alpha\left[\delta(x+a) + \delta(x-a)\right] \quad (\text{for } \alpha > 0)\ . \tag{3}$$

   As we did for the single delta-function well, define $\kappa = \sqrt{-2mE}/\hbar$.

(a) Since $V(x)$ is an even function, a stationary state wavefunction is either even or odd. Find the even bound state wavefunctions ($\psi(-x) = \psi(x)$) and a transcendental equation for κ. Don't bother normalizing the wavefunction. Give a graphical argument to count the number of allowed energies for even bound state wavefunctions (by "graphical," I mean show a sketch or computer-generated plot).

(b) Find the odd bound state wavefunctions ($\psi(-x) = -\psi(x)$) and a transcendental equation for κ. What condition must α and a satisfy for an odd bound state to exist?

(c) If an odd bound state exists, is the energy more negative for the even or odd bound state? Give a short justification.

4. **Numerical Determination of Energy Eigenvalue** *related to Griffiths 2.51*

Consider the potential

$$V(x) = -\frac{\hbar^2 a^2}{m}\frac{1}{\sqrt{1+a^2x^2}} \; . \tag{4}$$

Find the ground state energy using the numerical "wag the dog" method with Maple, as follows:

(a) Show that the Schrödinger equation for a bound state can be written in terms of dimensionless variables as

$$\frac{d^2\psi}{d\xi^2}(\xi) + \left(\frac{2}{\sqrt{1+\xi^2}} - \kappa^2\right)\psi(\xi) = 0 \; , \tag{5}$$

where κ is a positive constant.

(b) The ground state should be an even function, so if we ignore normalization, we can choose initial conditions $\psi(0) = 1$, $\psi'(0) = 0$. Starting with $\kappa^2 = 2.0$, enter the Schrödinger equation and initial conditions into Maple and solve the ODE numerically over the range $\xi = 0 - 10$. Then plot the solution. You may find the following Maple code helpful (note that we rename variables for ease of typing):

```
with(plots):
schr := diff(u(x),x$2)+(2/sqrt(1+x^2)-2.00)*u(x) = 0
init1 := u(0) = 1
init2 := (D(u))(0) = 0
psi := dsolve({init1, init2, schr}, numeric, range = 0 .. 10)
psiplot := odeplot(psi)
display(psiplot)
```

Attach a printout of your code with results. You should find a wavefunction with no nodes that blows up at large ξ.

(c) By decreasing your chosen value of κ^2 to 1.0, you should be able to get the "tail" of the wavefunction to flip over. Since the correct wavefunction should go to zero at large ξ, this means you have bracketed the correct eigenvalue for κ^2. Choose successively closer together values of κ^2 to find the eigenvalue down to three decimal places. What's the ground state energy? Attach a plot of your approximate ground state wavefunction.

(d) Now find the first excited state, which should have exactly one node at $x = 0$. In other words, take initial conditions $\psi(0) = 0$, $\psi'(0) = 1$ (ignoring normalization). Then for $x > 0$, $\psi(x) > 0$ (going to zero at $x \to \infty$. Use the procedure above starting with $\kappa^2 = 1$ to find the first excited state energy and attach a plot of the wavefunction.