PHYS-4601 Homework 3 Due 29 Sept 2016

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Expectation and Uncertainty**

   Consider an observable L with three eigenvalues +1, 0, and -1 and corresponding eigenstates $|+1\rangle, |0\rangle, |-1\rangle$. We have a system in state

   $$|\psi\rangle = \frac{1}{3}\Big(|+1\rangle + 2e^{i\beta}|0\rangle + 2|-1\rangle\Big) \ . \tag{1}$$

   (a) What is the probability of measuring each of the three eigenvalues of L?

   (b) Find the expectation value and uncertainty of a measurement of L.

   (c) Another observable A acts on the L eigenbasis as

   $$A|+1\rangle = \frac{1}{\sqrt{2}}|0\rangle \ , \quad A|0\rangle = \frac{1}{\sqrt{2}}\left(|+1\rangle + |-1\rangle\right) \ , \quad A|-1\rangle = \frac{1}{\sqrt{2}}|0\rangle \ . \tag{2}$$

   Find the expectation value and uncertainty of A in state $|\psi\rangle$.

   (d) Finally, show that the uncertainties of L and A satisfy the uncertainty principle in this state.

2. **Commutators and Functions of Operators**

   (a) Suppose $|\lambda\rangle$ is an eigenfunction of some operator $\mathcal{O}$, $\mathcal{O}|\lambda\rangle = \lambda|\lambda\rangle$. Consider the inverse operator $\mathcal{O}^{-1}$ defined such that $\mathcal{O}\mathcal{O}^{-1} = \mathcal{O}^{-1}\mathcal{O} = 1$. Show that $|\lambda\rangle$ is an eigenvector of $\mathcal{O}^{-1}$ with eigenvalue $1/\lambda$.

   (b) For any function $f(x)$ that can be written as a power series

   $$f(x) = \sum_n f_n x^n \ , \tag{3}$$

   we can define

   $$f(\mathcal{O}) = \sum_n f_n \mathcal{O}^n \ , \tag{4}$$

   where $\mathcal{O}^n$ denotes operating with $\mathcal{O}$ n times. Show that

   $$f(\mathcal{O})|\lambda\rangle = f(\lambda)|\lambda\rangle \ . \tag{5}$$

   Does this result hold if the power series includes negative powers?

   (c) For any three operators A, B, C, show that

   $$[A, BC] = [A, B]C + B[A, C] \ . \tag{6}$$

   (d) Then prove by induction that

   $$[A, B^n] = n[A, B]B^{n-1} \ , \tag{7}$$

   if $[A, B]$ commutes with B (for $n > 0$).

(e) Finally, show using (7) that $[p, f(x)] = -i\hbar df/dx$, where x and p are 1D position and momentum operators. Assume $f(x)$ can be written as a Taylor series.

3. **Gaussian Wavepacket Part I**

Here we take a first look at the Gaussian wavepacket in 1D, which is an important state in more than one physical system. In this problem, we will consider the state at a single instant $t = 0$, ignoring its time evolution. The state is

$$|\psi\rangle = \int_{-\infty}^{\infty} dx\ Ae^{-ax^2}|x\rangle\ . \tag{8}$$

Note that these results will be useful in future assignments.

(a) Find the normalization constant A. *Hint:* To integrate a Gaussian, consider its square. When you square it, change the dummy integration variable to y, then change the integral over $dxdy$ to plane polar coordinates. The textbook cover also has a formula for Gaussian integrals.

(b) Since the wavefunction is even, $\langle x\rangle = 0$. Find $\langle x^2\rangle$. *Hint:* You can get a factor of x^2 next to the Gaussian by differentiating it with respect to the parameter a.

(c) Write $|\psi\rangle$ in the momentum basis. *Hint:* If you have a quantity $ax^2 + bx$ somewhere, you may find it useful to write it as $a(x+b/2a)^2 - b^2/4a$ by completing the square. Then shift integration variables so it looks like you have a Gaussian again.

(d) Find $\langle p\rangle$ and $\langle p^2\rangle$ and show that this state saturates the Heisenberg uncertainty principle.