PHYS-4601 Homework 19 Due 28 Mar 2017

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Perturbation Theory vs Variational Principle**

   (a) Consider a particle moving in the 1D anharmonic oscillator potential

   $$V(x) = \frac{1}{2}m\omega^2 x^2 + gx^3 \tag{1}$$

   On the previous assignment, you found that the ground state energy remained $E_{gs} = \hbar\omega/2$ through first order in perturbation theory.

   Using the variational method, show that the true ground state energy of this potential is unbounded below (that is, if I give you any real number, demonstrate that the ground state energy is less than that number). We say that this potential is unstable and has no ground state. *Hint*: Think about a simple trial wavefunction that approximates a delta function in position.

   (b) *from Griffiths 7.5* Consider a Hamiltonian $H = H_0 + H_1$, where H_0 is exactly solvable and H_1 is small in some sense. Prove that first-order perturbation theory always overestimates the true ground state energy. That is, show that the ground state energy calculated in first-order perturbation theory is greater than (or equal to) the true ground state energy.

2. **WKB as $\hbar$ Expansion** *Based on Griffiths 8.2*

   In this problem, you'll derive the WKB wavefunction in regions where $E > V(x)$. We will use the 1D Schrödinger equation in the form

   $$\frac{d^2\psi}{dx^2} = -\frac{p(x)^2}{\hbar^2}\psi\ , \quad p(x) = \sqrt{2m(E - V(x))}\ . \tag{2}$$

   (a) Begin by writing the wavefunction as $\psi(x) = \exp[if(x)/\hbar]$ for some complex function f (note that this is completely general). Use the Schrödinger equation (2) to find a second order differential equation for f.

   (b) Now treat $\hbar$ as a small parameter, expanding $f(x) = f_0(x) + \hbar f_1(x) + \cdots = \sum_n \hbar^n f_n(x)$. Then write your differential equation from part (a) as a series in $\hbar$. Find the differential equations that come from the $\hbar^0$, $\hbar^1$, and $\hbar^2$ terms; these should vanish separately as $\hbar \to 0$.

   (c) Find f_0 and f_1 in terms of $p(x)$, and use those to derive the WKB form of the wavefunction. *Hint:* Recall that $\ln z = \ln|z| + i\theta$, where the complex $z = |z|e^{i\theta}$.

   (d) Describe how you would change this procedure for regions where $V(x) > E$.

3. **Uniform Gravitational Field** *parts of Griffiths 8.5 and 8.6*

   Consider a ball of mass m that feels a uniform gravitational acceleration g in the $-x$ direction, as by the surface of the earth. Assume that the surface of the earth is at $x = 0$ and forms an infinite potential barrier.

   (a) First, write down what the potential energy is as a function of x.

(b) Use the WKB approximation to find the allowed energies of the bouncing ball. Find the approximate ground state and first excited state energies in Joules to two significant digits for a neutron (mass $m = 1.7 \times 10^{-27}$ kg). This can actually be measured for ultracold neutrons.

(c) The *exact* solution of the Schrödinger equation is given by the Airy function

$$\psi(x) = C\mathrm{Ai}\left[\left(\frac{2m^2 g}{\hbar^2}\right)^{1/3}\left(x - \frac{E}{mg}\right)\right] , \tag{3}$$

where C is a normalization constant and E is quantized so $\psi(0) = 0$. Denote the zeros of $\mathrm{Ai}(z)$ by a_k ($k = 1, 2, \cdots$ with $|a_1| < |a_2| < \cdots$) and find the energy eigenvalues in terms of the a_k. What are the ground and first excited state energies for a neutron? You will need to look up values of a_k at the Digital Library of Mathematical Functions (DLMF) at `http://dlmf.nist.gov/9.9`.

(d) Show that the energy eigenvalues match the WKB result in the limit of large quantum number. *Hint:* You can use the asymptotic form of the Airy function itself (either in Griffiths or in the DLMF) or that of the zeros (from the DLMF).

4. **Ionizing an Atom** *from Griffiths 8.16*

Imagine a hydrogen atom in a small electric field; the electron feels a linear potential from the field, which eventually becomes less than the ground state energy, so it can tunnel out of the atom. In this problem, consider a simple 1D model of this system, with potential

$$V(x) = \begin{cases} \infty , & x < -a \\ -V_0 , & -a < x < 0 \\ -\alpha x , & x > 0 \end{cases} . \tag{4}$$

(a) Suppose the square well is very deep, so $V_0 \gg \hbar^2/ma^2$. In the absence of the electric field ($\alpha = 0$), what is the approximate ground state energy E? If the electron were a classical particle with this kinetic energy, what would be its speed? *Hint:* You can think of this as the energy of the first odd eigenfunction of a finite square well of width $2a$ or you can approximate the potential as nearly an infinite square well.

(b) Show that the lifetime of the atom in the presence of the field is $\ln \tau = A|E|^{3/2} + B$, where A and B are constants. Then find A and B (you may need your results from part (a)).