PHYS-4601 Homework 17 Due 2 Mar 2017

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

Useful information for this assignment:
From our discussion of the free electron gas, the number of states of a free particle with wavevector magnitude between k and $k + dk$ in a volume V is

$$\frac{g}{8}\frac{4\pi k^2 dk}{\pi^3/V} = \frac{gV}{2\pi^2}k^2 dk \ , \tag{1}$$

where g is the number of spin or polarization states. This is one octant of a spherical shell of radius k and thickness dk with one state per π^3/V. This number of states is true for any type of free particle, relativistic or not. *You may set Boltzmann's constant $k_B = 1$ for this assignment.*

1. **An Integral, Gamma, and Zeta Functions**

   The gamma function and Riemann zeta function are defined as

   $$\Gamma(z) = \int_0^\infty dx\, x^{z-1}e^{-x} \text{ and } \ \zeta(s) = \sum_{n=1}^{\infty} n^{-s} \ . \tag{2}$$

   For use on the rest of the assignment, prove that

   $$\int_0^\infty dx\, \frac{x^{s-1}}{e^x - 1} = \Gamma(s)\zeta(s) \ . \tag{3}$$

   In the remaining problems, use equation (3) when appropriate and note that $\Gamma(n+1) = n!$ for n a non-negative integer. *Hint:* expand the fraction as a geometric series $\sum_{n=0}^{\infty} x^n = 1/(1-x)$.

2. **Bose-Einstein Condensation** *from Griffiths 5.29*

   Consider the Bose-Einstein distribution for spin-0 bosons. As in equation (1), we can usually approximate the sum over states as an integral over wavevectors so that the number density and energy density are

   $$n = \frac{1}{2\pi^2}\int_0^\infty dk\, k^2 f(x) \ , \quad \rho = \frac{\hbar^2}{4m\pi^2}\int_0^\infty dk\, k^4 f(x) \ , \quad x = \frac{1}{T}\left(\frac{\hbar^2 k^2}{2m} - \mu\right) \ , \tag{4}$$

   where f is the Bose-Einstein distribution $f(x) = 1/(e^x - 1)$. Note that a spin-0 boson has $g = 1$ spin state.

   If there is a fixed number density n of bosons, the chemical potential μ is a function of T. $\mu = 0$ at a fixed critical temperature T_c; $\mu < 0$ for higher temperatures, and the integral approximation above fails for lower temperatures because all the bosons go into the ground state (this is *Bose-Einstein condensation*). Find T_c as a function of n. You may write your answer in terms of gamma and zeta functions.

3. **Proton Chemical Potential in Cosmology**

   For much of the history of the universe, protons are nonrelativistic and in thermal equilibrium with photons. During that time period, the number densities of protons n_B and photons n_γ are determined by the Fermi-Dirac and Bose-Einstein distributions with the same temperature.

(a) Photons with $g = 2$ polarization states have single particle energies given by $\epsilon = c\hbar k$. Assuming zero chemical potential, show that the number density of photons at temperature T is

$$n_\gamma = \frac{2\zeta(3)}{\pi^2}\left(\frac{T}{\hbar c}\right)^3 . \tag{5}$$

Hint: Remember, photons obey Bose-Einstein statistics.

(b) Now consider nonrelativistic protons. Including the rest mass energy, a single particle state has energy $\epsilon = mc^2 + \hbar^2 k^2/2m$. Assume that $0 < \mu < mc^2$ and find the number density n_p of protons when $T \ll (mc^2 - \mu)$. *Hint:* Argue that the 1 in the denominator of the distribution function is negligible in this limit.

(c) The ratio $\eta = n_p/n_\gamma$ is known as the *baryon-to-photon* ratio in cosmology (ignoring the small proportion of elements besides hydrogen), and it is constant over the period of the universe we consider with a value of $\eta \approx 10^{-10}$. Find the chemical potential μ of the protons in terms of η, the temperature T, the proton mass m, and physical constants. You may ignore constants of order unity.