PHYS-4601 Homework 14 Due 2 Feb 2017

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **2-Qbit Gates**

   Consider a 2 qbit system. Choose a basis for the 2 qbit Hilbert space and use it for all parts of this problem.

   (a) Write the CNOT gate operator as a matrix in that basis and show that it is unitary.

   (b) Consider the 1 qbit gate NOT acting only on the first qbit of our two. Write this gate (call it NOT_1) as a matrix in your 2-qbit basis.

   (c) *inspired by Blümel 7.5.4* We can create a new quantum gate G by first acting with the NOT_1 and then CNOT. Give an example of an input 2-qbit state that can be factorized (that is, written as $|\psi\rangle_1|\phi\rangle_2$ for some 1-qbit states $|\psi\rangle$, $|\phi\rangle$) that is turned into an entangled state by G ($G(|\psi\rangle_1|\phi\rangle_2)$ cannot be factorized).

2. **Cloning Means FTL Communication** *based on a problem by Wilde*

   Suppose that Alice and Bob are at two ends of an EPR/Bell experiment. In other words, they are at rest with respect to each other and separated by 5 lightyears, and each receives one of a pair of entangled electrons with total spin state $s = 0$ simultaneously (in their common rest frame). By prior agreement, Alice measures either the S_z or S_x spin of her electron as soon as she receives it, but Bob does not know which spin she measures.

   After Alice's measurement (in their rest frame time), Bob's electron is in some state $|\psi\rangle_B$. Suppose, in contradiction to the no-cloning theorem, Bob can clone his electron's state onto a large number N of other electrons. (For example, Bob can do some quantum operation that takes his $N+1$ electrons from state $|\psi\rangle_B|\uparrow\rangle_1 \cdots |\uparrow\rangle_N$ to state $|\psi\rangle_B|\psi\rangle_1 \cdots |\psi\rangle_N$.) What measurement(s) can Bob do on his extra N electrons that will tell him with great certainty whether Alice measured the S_z or S_x spin of her electron? Explain your answer. (Note that Bob can accomplish his measurement before Alice can tell him her measurement choice, so they can establish faster-than-light communication in this way. This is a good reason for the no-cloning theorem!)

3. **The Density Matrix/Operator**

   A system in a well-defined quantum state $|\psi\rangle$ (ie, a linear superposition of some basis kets) is said to be in a *pure state*. On the other hand, suppose you have a friend in a laboratory that produces a quantum system in some specific state, but you don't know what state it is in. You know that there is a probabilty P_n that your friend has made state $|\psi_n\rangle$. From your perspective, the system is in a *mixed state*, which is described by the classical probability of each quantum state. We can describe this mixed state by a *density operator* (also known as the *density matrix*)

   $$\rho \equiv \sum_n P_n \, |\psi_n\rangle\langle\psi_n| \ , \tag{1}$$

   which represents your ignorance of the true quantum state of the system. A density operator can be used to represent a thermal state of a quantum system (with probabilities given by the Boltzmann factor). A pure state is represented by a density operator where $P_n = 1$ for some specific state $|\psi_n\rangle$ with all other $P_n = 0$.

(a) Suppose the system is the spin of an electron, and there is a 50% probability each that your friend has produced the spin either up along z or up along x. Write the density matrix first in terms of the states $|\uparrow_z\rangle, |\uparrow_x\rangle$ and then the basis states $|\uparrow_z\rangle, |\downarrow_z\rangle$. *Hint:* the S_x eigenstates are given in §4.4 of Griffiths.

(b) Another way mixed states can arise is through entanglement. For two particles in the pure state $|\psi\rangle_{1,2}$, define the two-particle density operator as usual. Then the density operator for the first particle can be defined as

$$\rho_1 = \sum_i {}_2\langle e_i|\rho|e_i\rangle_2 \ , \tag{2}$$

where $|e_i\rangle$ is a basis of single-particle states. Suppose two electrons are in the $|s = 0\rangle_{1,2}$ total spin state. Find the density operator of the first electron. (This definition is useful in quantum information theory.)