PHYS-4601 Homework 11 Due 5 Jan 2016

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Hyperon Decay**

   A spin-3/2 Ω^- hyperon at rest decays into a spin-1/2 Λ hyperon and a spin-0 K^- meson. (These are subatomic particles.)

   (a) What are the allowed values of the orbital angular momentum quantum number ℓ of the final particles, consistent with conservation of (total) angular momentum? Note that the total spin of the two product particles is $s = 1/2$ (ie, just the spin of the Λ).

   (b) Assume the Ω^- is in a spin state with S_z eigenvalue $+3\hbar/2$. Write the most general possible combined orbital angular momentum and spin state $|\psi\rangle$ of the final particles. Express your answer as a superposition of basis states of the form $|\ell, m\rangle|s, m_s\rangle$. *Hint:* For each value of ℓ you found in part (a), look up the Clebsch-Gordon coefficients to write a $j = 3/2, m = 3/2$ state in terms of $|\ell, m\rangle|s, m_s\rangle$ states. Then write a superposition of the formulas for each possible value of ℓ.

   (c) Substitute the orbital part of the angular momentum state you found in part (b) above with the appropriate spherical harmonic, where θ, ϕ represent the direction of motion of the K^- particle. Then the normalization of the state can be written as

   $$\langle\psi|\psi\rangle = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, \sin\theta \, P(\theta, \phi) \ ,$$

   where $P(\theta, \phi)$ is the probability density for the angular distribution of the K^-. Find $P(\theta, \phi)$. *Hint:* Your state should take the form $|\psi\rangle \simeq \psi_+(\theta, \phi)|+\rangle + \psi_-(\theta, \phi)|-\rangle$. Remember that $|+\rangle$ and $|-\rangle$ are orthogonal.

2. **Center of Mass Frame and Reduced Mass**

   In class, we treat the hydrogen atom as if it is an electron moving around a stationary proton. Of course, that can't be, since it violates conservation of momentum. What happens, of course, is that the proton hardly moves in the center of mass rest frame. However, it turns out that we can always describe a system of two particles in terms of a single particle. In this problem, consider two particles of masses m_1 and m_2.

   (a) In quantum mechanics, the kinetic energy is given by a Laplacian operator. Consider the 1D case for simplicity. Then the kinetic Hamiltonian is

   $$H = -\frac{\hbar^2}{2m_1}\frac{d^2}{dx_1^2} - \frac{\hbar^2}{2m_2}\frac{d^2}{dx_2^2} \ , \tag{1}$$

   where x_1 is the first particle's position and x_2 is the second particle's position. Show that this kinetic Hamiltonian can be written as

   $$H = -\frac{\hbar^2}{2M}\frac{d^2}{dX^2} - \frac{\hbar^2}{2\mu}\frac{d^2}{dx^2} \ , \tag{2}$$

where $M = m_1 + m_2$ is the total mass, $\mu = m_1 m_2 / M$ is the reduced mass, $X = (m_1 x_1 + m_2 x_2)/M$ is the center of mass position, and $x = x_1 - x_2$ is the relative position.

The proof is essentially the same for the 3D Laplacian, and we then set the center of mass momentum to zero by choice of reference frame. Therefore, when we study the hydrogen atom, we are really using the reduced mass of the electron, which is nearly the electron mass because the proton is so much heavier than the electron.

(b) Imagine an atom made of an electron and a deuteron (nucleus made of a proton and neutron); this is a deuterium atom. Deuterium atoms are exactly like hydrogen atoms (in terms of energy eigenvalues) except the proton mass is replaced by the deuteron mass. Find the fractional difference in ground state energies $(E_D - E_H)/E_H$, where E_D is the deuterium ground state energy and E_H is the hydrogen ground state energy. First, give your answer in terms of the electron, proton, and deuteron masses to $\mathcal{O}(m_e/m_p, m_e/m_d)$ and then numerically using $m_e = 0.511$ MeV/c^2, $m_p = 938$ MeV/c^2, $m_d = 1876$ MeV/c^2 (give your answer to three significant digits).

3. **Comparing Expectation Values**

   (a) *based on Griffiths 4.13* Find $\langle r^2 \rangle$ for the ground state of a hydrogen-like atom (a single electron moving in a central Coulomb potential) in terms of the Bohr radius. Find the ratio of this result between hydrogen to that for a helium ion, which has a single electron orbiting a nucleus of charge $+2e$. (In other words, find $\langle r^2 \rangle_H / \langle r^2 \rangle_{He^+}$.) What does this mean about the comparative "size" of these two atoms?

   (b) Now find the ratio of $\langle r^2 \rangle$ for the $n = 2, \ell = 1, m = 0$ state of hydrogen to the $n = 2, \ell = 0, m = 0$ state.

   (c) Finally, find the ratio of $\langle z^2 \rangle$ for the $n = 2, \ell = 1, m = 0$ state of hydrogen to the $n = 2, \ell = 0, m = 0$ state. *Hint:* You can find $\langle z^2 \rangle$ for the $n = 2, \ell = 0, m = 0$ state by using symmetry arguments and your work from part (b).

4. **Classical Limit of Hydrogen** *from Griffiths 4.46*

   In this problem, consider states of the hydrogen atom with principle quantum number n and maximum orbital angular momentum $\ell = n - 1$.

   (a) Show that the radial wavefunction is

   $$R(r) = A r^{n-1} e^{-r/na} \ , \tag{3}$$

   where A is a normalization constant, and find A.

   (b) Find $\langle r \rangle$ and $\langle r^2 \rangle$.

   (c) Find the uncertainty σ_r and show that $\sigma_r / \langle r \rangle \propto 1/\sqrt{n}$ for large n. How does this correspond to the idea that "classical physics corresponds to high excitation numbers?"

5. **Electron Capture**

   Nuclear decay by *electron capture* occurs when a proton in the nucleus and an electron combine to form a neutron. The rate of this decay is proportional to the probability that the electron is found inside the nucleus. Find this probability for an electron in the $n = 1$ state of an atom with Z protons and A total neutrons plus protons. Assume(1) the nucleus is a sphere of radius $A^{1/3} L$, where L is some typical nuclear length, (2) the nuclear radius is much smaller than the Bohr radius of the atom (*Hint*: use this to simplify integrals *before* integrating), (3) the wavefunction of this electron is unaffected by the presence of any other electrons in the atom (this is generally not a good approximation, but use it anyway). State your answer in terms of the Bohr radius a of hydrogen, Z, A, and L.