PHYS-4601 Homework 1 Due 15 Sept 2016

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Dirac Notation on the Circle**

   Consider the Hilbert space of L^2 functions on the interval $0 \le x \le 2\pi R$ with periodic boundary conditions.

   (a) Show that the complex exponentials $|e_n\rangle \simeq e^{inx/R}/\sqrt{2\pi R}$ for n any integer form an orthonormal set. As it turns out, they make a complete orthonormal basis (but you do not have to prove that).

   Carry out the following calculations *without doing any integrals.*

   (b) Calculate the inner product of $|f\rangle \simeq f(x) = \cos^3(x/R)$ and $|g\rangle \simeq g(x) = \sin(3x/R)$.

   (c) Find the inner product of $|f\rangle$ and $|g\rangle$ from part (b) with $|h\rangle \simeq h(x) = \sin(3x/R + \theta)$.

   (d) $|f\rangle, |g\rangle, |h\rangle$ are not normalized. Find their norms.

2. **Dual Vectors and Change of Basis**

   Consider a 3-dimensional complex column vector space, which has the usual orthonormal basis

$$|e_1\rangle \simeq \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \ , \ |e_2\rangle \simeq \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \ , \ |e_3\rangle \simeq \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} . \tag{1}$$

   In that basis, the vectors $|f_i\rangle$ $(i = 1, 2, 3)$ can be written as

$$|f_1\rangle \simeq \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \ , \ |f_2\rangle \simeq \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \ , \ |f_3\rangle \simeq \frac{1}{\sqrt{6}} \begin{bmatrix} i \\ -i \\ -2i \end{bmatrix} . \tag{2}$$

   (a) Write the $|f_i\rangle$ as linear superpositions of the $|e_i\rangle$ basis vectors.

   (b) Show that the $|f_i\rangle$ are normalized and mutually orthogonal, so they form a complete orthonormal basis (distinct from the set of $|e_i\rangle$).

   (c) Write the associated dual vectors $\langle f_i|$ as row vectors in the $\{\langle e_i|\}$ basis.

   (d) Write the $|e_i\rangle$ vectors as linear superpositions of the $|f_i\rangle$. Use your result to do a change of basis for this Hilbert space by writing the $|e_i\rangle$ vectors as column vectors in the $\{|f_i\rangle\}$ basis. *Hint:* You can solve a system of linear equations or use a similarity transformation, but it is much easier if you use inner products as discussed in the notes.

3. **Superposition of States**

   Suppose $|\psi\rangle$ and $|\phi\rangle$ are two normalized state vectors, and so is $|\alpha\rangle = A(3|\psi\rangle + 4|\phi\rangle)$.

   (a) Find the normalization constant A in the case that

      i. $\langle\psi|\phi\rangle = 0$.

      ii. $\langle\psi|\phi\rangle = i$.

      iii. $\langle\psi|\phi\rangle = e^{i\pi/6}$.

(b) In the case $\langle\psi|\phi\rangle = i$, use the *Gram-Schmidt procedure* described in Griffiths problem A.4 to find the part of $|\alpha\rangle$ orthogonal to $|\psi\rangle$. Verify that it is orthogonal by taking the inner product.

(c) Now suppose that $\langle\psi|\phi\rangle = 0$ and define a new state $|\beta\rangle = B(4e^{-i\theta}|\psi\rangle + 3e^{i\theta}|\phi\rangle)$ for some angle θ. Find the normalization constant B and $\langle\alpha|\beta\rangle$ (you make assume that the normalization constants are positive and real).

4. **Fourier-Bessel Series**

Bessel functions (of the first kind) $J_m(x)$ of order m solve the Bessel differential equation

$$x^2\frac{d^2J_m}{dx^2} + x\frac{dJ_m}{dx} + (x^2 - m^2)J_m = 0 \tag{3}$$

and satisfy $J_m(0) = 0$ for $m > 0$. By using the Bessel differential equation, it is possible to show that the functions $J_m(k_n x)$ defined for $0 \le x \le 1$ are orthogonal with respect to the real inner product

$$\langle f|g\rangle = \int_0^1 dx\, x f(x)g(x) \text{ for } |f\rangle \simeq f(x)\ ,\ |g\rangle \simeq g(x) \tag{4}$$

whenever k_n are zeros of the Bessel function $J_m(k_n) = 0$. In other words, with suitably chosen normalization constants A_n, the vectors $|n\rangle \simeq A_n J_m(k_n x)$ $(n = 1, 2, \cdots)$ form an orthonormal set.

(a) These orthonormal functions are defined on $0 \le x \le 1$ with *Dirichlet boundary conditions* $f(0) = f(1) = 0$. Show that the set of functions on $0 \le x \le 1$ with Dirichlet boundary conditions is a vector space.

(b) Our orthonormal vectors $|n\rangle$ form a basis. That means that any vector in our vector space can be written as a sum

$$|f\rangle = \sum_n c_n|n\rangle\ . \tag{5}$$

Show that

$$c_n = A_n \int_0^1 dx\, x f(x) J_m(k_n x)\ . \tag{6}$$