Interactions With Charges

- The basic set up
  - The Hamiltonian, in a perturbation theory sense, is

$$H = H_0 + H_{EM} + H_{int}$$

+ $H_0 = \vec{p}^2/2m + V(\vec{x})$, where $V(\vec{x})$ may be electrostatic
+ H_{EM} is as above + $H_{int} = \frac{-q}{2m}\left(\vec{p}\cdot\vec{A}(\vec{x}) + \vec{A}(\vec{x})\cdot\vec{p}\right)$

+ We assume that the $\vec{A}^2$ term is negligible usually.

- The Hilbert space has a factorized basis as you'd expect

$$|\psi_{tot}\rangle = |\psi_{particle}\rangle\, |n_{\lambda_1}(\vec{k}_1), \ldots\rangle$$

- We want to know the transition rate for the system to make a transition

$$|\psi_i\rangle |n_{\lambda_1}(\vec{k}_1), \ldots\rangle \rightarrow |\psi_f\rangle |n_{\lambda_1}(\vec{k}_1) \pm 1, \ldots\rangle$$

ie, for the charged particle to emit or absorb a photon (photon states otherwise same)

+ We're looking at single photon events. Multiple photons are higher order in perturbation theory.
+ Need to think a moment about perturbation theory. We have time dependence in $\vec{A}(\vec{x},t)$, so we are using the Heisenberg picture for photon states. But we can stick to the Schrödinger picture for the other system + use time-dependent perturbation theory.

+ We will need

$$\mathcal{M} = \langle\psi_{tot,f}|H_{int}|\psi_{tot,i}\rangle = \sqrt{\frac{\hbar}{\epsilon_0}}\,\frac{1}{L^{3/2}}\sum_{\lambda\vec{k}}\frac{1}{\sqrt{2\omega_{\vec{k}}}}\left(\frac{-q}{2m}\right)\Big[\langle\psi_f|\left(\vec{p}\cdot\hat{\epsilon}_\lambda e^{i\vec{k}\cdot\vec{x}} + e^{i\vec{k}\cdot\vec{x}}\,\vec{p}\cdot\hat{\epsilon}_\lambda\right)|\psi_i\rangle e^{-i\omega_k t}$$
$$\langle n_{\lambda_1}(\vec{k}_1)\pm 1, \ldots|a_\lambda(\vec{k})|n_{\lambda_1}(\vec{k}_1), \ldots\rangle$$
$$+ \langle\psi_f|\left(\vec{p}\cdot\hat{\epsilon}_\lambda e^{-i\vec{k}\cdot\vec{x}} + e^{-i\vec{k}\cdot\vec{x}}\,\vec{p}\cdot\hat{\epsilon}_\lambda\right)|\psi_i\rangle\langle n_{\lambda_1}(\vec{k}_1)\pm 1, \ldots|a_\lambda^\dagger(\vec{k})|n_{\lambda_1}(\vec{k}_1), \ldots\rangle e^{+i\omega_k t}$$

+ The photon parts set $\vec{k}=\vec{k}_1$, $\lambda=\lambda_1$. We also set $\vec{k}\cdot\vec{x} \ll 1$ for "typical sizes" of the system as a simplifying assumption (allows us to neglect recoil, etc). Set $e^{i\vec{k}\cdot\vec{x}}=1$.

+ Our amplitude becomes

$$\mathcal{M} = -\frac{q}{m}\sqrt{\frac{\hbar}{\epsilon_0}}\ \frac{1}{L^{3/2}}\ \frac{1}{\sqrt{2\omega_{\vec{k}_1}}}\ \sqrt{n_{\lambda_1}(\vec{k}_1)}\ \langle \psi_f | \vec{p}\cdot\hat{\epsilon}_{\lambda_1} | \psi_i \rangle\, e^{-i\omega_{\vec{k}_1}t}$$ <u>absorption</u>

$$\text{or} = -\frac{q}{m}\sqrt{\frac{\hbar}{\epsilon_0}}\ \frac{1}{L^{3/2}}\ \frac{1}{\sqrt{2\omega_{\vec{k}_1}}}\ \sqrt{n_{\lambda}(\vec{k}_1)+1}\ \langle \psi_f | \vec{p}\cdot\hat{\epsilon}_{\lambda} | \psi_i \rangle\, e^{+i\omega_{\vec{k}_1}t}$$ <u>emission</u>

- We have something of the form $V_{fi}\,e^{\pm i\omega t}$, like in time-dependent perturbation theory

  + Fermi's golden rule (see Homework) tells us that the transition rate to each final state

$$\Gamma = \frac{\text{prob.}}{\text{time}} = \frac{2\pi}{\hbar^2}\,|V_{fi}|^2\,\delta\!\left(\frac{E_f-E_i}{\hbar} \mp \omega_{\vec{k}_1}\right) \quad \begin{cases}\text{absorption}\\ \text{emission}\end{cases}$$

  We see energy conservation b/c $\hbar\omega_{\vec{k}_1}$ is the photon energy

  + Absorption + stimulated emission rate $\propto$ # of photons in same state. This means stimulated emission rate is $N\times$ spontaneous emission rate. For large N, you can treat absorption + stimulated emission w/ classical EM fields.

  How lasers work.

- Decays + Spontaneous Emission.

  We have $|\psi_i\rangle|0\rangle \rightarrow |\psi_f\rangle|\vec{k},\lambda\rangle$

  - The decay rate into a single momentum/polarization is

$$\Gamma_{\vec{k},\lambda} = \frac{2\pi}{\hbar\epsilon_0}\,\frac{q^2}{m^2}\,\frac{1}{2ckL^3}\,|\langle\psi_f|\,\vec{p}\cdot\hat{\epsilon}_\lambda|\psi_i\rangle|^2\,\delta\!\left(\frac{E_i-E_f}{\hbar}-ck\right)$$

  - Let's note that, for $H_0 = \frac{\vec{p}^2}{2m} + V(\vec{x})$, $\vec{p} = \frac{-im}{\hbar}[\vec{x}, H_0]$

  + $\Rightarrow$ $\langle\psi_f|\vec{p}\cdot\hat{\epsilon}_\lambda|\psi_i\rangle = \frac{i}{\hbar}m(E_f-E_i)\,\langle\psi_f|\vec{x}\cdot\hat{\epsilon}_\lambda|\psi_i\rangle$

+ That means

$$\Gamma_{\vec{k},\lambda} = \left(\frac{\pi}{\epsilon_0}\right)\left(\frac{E_i-E_f}{\hbar^2}\right)\frac{1}{L^3}\,|\langle\psi_f|\vec{p}\cdot\hat{\epsilon}_\lambda|\psi_i\rangle|^2\,\delta\!\left(\frac{E_i-E_f}{\hbar}-kc\right)$$

where $\vec{p}$ is the electric dipole moment of the transition.

- Now take a spherically symmetric system (like Hydrogen). Photons have 1 unit of spin $\Rightarrow$ selection rules for transitions between different angular states
  - + $[L_z, z] = 0 \Rightarrow \langle l', m'|[L_z,z]|l,m\rangle = (m'-m)\langle l',m'|z|l,m\rangle = 0$

    So $m'=m$ or $\langle p_z\rangle = 0$.
  - + Similarly, $m'-m=\pm 1$ or $\langle p_x\rangle = \langle p_y\rangle = 0$.
  - + With a bit more work, you can see $l'-l=\pm 1$
  - + That means the hydrogen state $|2,0,0\rangle$ is metastable.

    It can only decay at higher order in $\vec{k}$ or perturbation theory

    Quadrupole
- Finally, the total decay rate sums over momenta and polarizations

$$\Gamma = \sum_{\vec{k},\lambda}\Gamma_{\vec{k},\lambda}$$

+ Recall that the momenta are $\vec{k} = \left(\frac{2\pi n_x}{L}, \frac{2\pi n_y}{L}, \frac{2\pi n_z}{L}\right)$ for periodic b.c.

so $\frac{1}{L^3}\sum_{\vec{k}} \to \int\frac{d^3\vec{k}}{(2\pi)^3}$

+ Therefore

$$\Gamma = \frac{\pi e^2}{\epsilon_0}\,\frac{E_i-E_f}{\hbar^2}\,\frac{1}{c}\left(\frac{E_i-E_f}{\hbar c}\right)^2\int\frac{d\Omega}{(2\pi)^3}\sum_\lambda|\langle\psi_f|\vec{x}\cdot\hat{\epsilon}_\lambda|\psi_i\rangle|^2$$

+ Last trick: To simplify, note $\sum_\lambda \hat{\epsilon}_{\lambda,i}\hat{\epsilon}_{\lambda,j}$ projects $\perp \hat{k}$, so $= \delta_{ij} - \hat{k}_i\hat{k}_j$. As in the HW on hyperfine structure, the angle integral of $\hat{k}_i\hat{k}_j$ is $\frac{4\pi}{3}\delta_{ij}$

$$\Gamma = \frac{4\alpha}{3}\frac{\omega_0^3}{c^2}|\langle\psi_f|\vec{x}|\psi_i\rangle|^2$$ after simplification. (α = fine structure, $\hbar\omega_0 = E_i - E_f$)

+ You'll finish an example for hydrogen on HW.