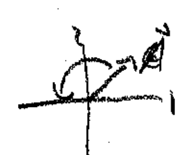
Introduction to Quantum Electrodynamics

be careful of units in optional reading

- Photons: Quantum units of free EM waves
  - What do we know about how EM fields + waves work?
    - The "physical" or observable quantities are $\vec{E}$ + $\vec{B}$
      + Satisfy Maxwell's equations (with no sources)
        $$\vec{\nabla}\cdot\vec{E}=0,\quad \vec{\nabla}\cdot\vec{B}=0,\quad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t},\quad \vec{\nabla}\times\vec{B}=\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$$
      + The total energy (Hamiltonian) is
        $$H=\frac{1}{2}\int d^3\vec{x}\left(\epsilon_0\vec{E}^2+\frac{1}{\mu_0}\vec{B}^2\right)=\frac{\epsilon_0}{2}\int d^3\vec{x}\left(\vec{E}^2+c^2\vec{B}^2\right)$$
      + How do you get Maxwell's eqns from that?
    - We know you can use potentials to describe the fields
      $$\vec{E}=-\vec{\nabla}\Phi-\frac{\partial\vec{A}}{\partial t},\quad \vec{B}=\vec{\nabla}\times A$$
      + There is "gauge freedom" to change $\Phi\to\Phi-\frac{\partial\lambda}{\partial t}$, $\vec{A}\to\vec{A}+\vec{\nabla}\lambda$.
      + For radiation, it is useful to choose $\Phi=0$ ("temporal gauge")
        Gauss's law tells us that $\vec{\nabla}\cdot\vec{A}=0$ with no charges
      + Then $\vec{E}=-\dot{\vec{A}}$ looks like a momentum.
        - Gauss law = constraint
        - Next 2 automatically satisfied
        - Ampère's from Hamilton's eqns.
    - Waves are described by Fourier modes
      + Each mode is $\vec{A}\propto\left(a_1(\vec{k})\hat{\epsilon}_1+a_2(\vec{k})\hat{\epsilon}_2\right)e^{i\vec{k}\cdot\vec{x}-i\omega t}+c.c.$
      + The two polarization vectors satisfy $\vec{k}\cdot\hat{\epsilon}=0$. Transverse wave
        We have chosen linear polarization, but $\hat{\epsilon}_\pm=\frac{1}{\sqrt{2}}(\hat{\epsilon}_1\pm i\hat{\epsilon}_2)$ = circularly polarized
      + The "temporal" polarization doesn't exist b/c $\Phi=0$ and Gauss's law means there is no longitudinal polarization.
      + The last Maxwell eqn (Ampère's law) says $\omega^2=c^2|\vec{k}|^2$

Circular polarization carries angular momentum



— Implementing EM waves in QM

- We have something a lot like a harmonic oscillator
  - $$H = \frac{1}{2}\int d^3\vec{x}\left[\vec{\Pi}^2(\vec{x}) + \epsilon_0 c^2 |\vec{\nabla}\times\vec{A}(\vec{x})|^2\right] \sim \frac{P^2}{2m} + \frac{1}{2}m\omega^2 x^2$$
  - To see this more clearly, let $\vec{A}(\vec{x},t) \equiv \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \vec{A}(\vec{k},t) e^{i\vec{k}\cdot\vec{x}}$ & similarly for $\vec{\Pi}$ w/ $\vec{A}(-\vec{k}) = \vec{A}^{*\dagger}(\vec{k})$
  - Then

$$\int d^3\vec{x}\, |\vec{\nabla}\times\vec{A}(\vec{x})|^2 = \int d^3\vec{x} \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \int \frac{d^3\vec{k}'}{(2\pi)^{3/2}} - (\vec{k}'\times\vec{A}(\vec{k}'))\cdot(\vec{k}\times\vec{A}(\vec{k}))\, e^{i(\vec{k}+\vec{k}')\cdot x}$$

$$= -\int d^3\vec{k} \int d^3\vec{k}' \,(\vec{k}'\times\vec{A}(\vec{k}'))\cdot(\vec{k}\times\vec{A}(\vec{k}))\,\delta^3(\vec{k}+\vec{k}')$$

$$= \int d^3\vec{k}\, |\vec{k}\times\vec{A}(\vec{k})|^2 = \int d^3\vec{k}\, k^2 |\vec{A}(\vec{k})|^2$$ by vector identities

So

$$H = \frac{1}{2}\int d^3\vec{k} \left[|\Pi(\vec{k})|^2 + \epsilon_0 c^2 k^2 |\vec{A}(\vec{k})|^2\right] = \text{one SHO per wavenumber!}$$

- Let's be more careful: Better to work in finite volume of length L
  - $$\vec{A}(\vec{x},t) = \sqrt{\frac{\hbar}{\epsilon_0}} \frac{1}{L^{3/2}} \sum_\lambda \sum_{\vec{k}} \frac{1}{\sqrt{2\omega_{\vec{k}}}} \hat{\epsilon}_\lambda(\vec{k})\left[a_\lambda(\vec{k}) e^{i(\vec{k}\cdot\vec{x} - \omega_{\vec{k}} t)} + a_\lambda^\dagger(\vec{k}) e^{-i(\vec{k}\cdot\vec{x} - \omega_{\vec{k}} t)}\right]$$

    We have cleverly introduced a normalization factor without explaining why. Define $\omega_{\vec{k}} = c|\vec{k}|$, λ = polarization
  - Note that

$$\vec{E} = \sqrt{\frac{\hbar}{\epsilon_0}} \frac{1}{L^{3/2}} \sum_\lambda \sum_{\vec{k}} \sqrt{\frac{\omega_{\vec{k}}}{2}}\, \hat{\epsilon}_\lambda(\vec{k})\left[i\, a_\lambda(\vec{k}) e^{i(\vec{k}\cdot\vec{x} - \omega_k t)} - i\, a_\lambda^\dagger(\vec{k}) e^{-i(\vec{k}\cdot\vec{x} - \omega_k t)}\right]$$

  - If you go through all the Fourier transforms carefully,

$$H = \sum_{\vec{k}} \frac{\hbar\omega_{\vec{k}}}{2} \sum_\lambda \left(a_\lambda^\dagger(\vec{k})\, a_\lambda(\vec{k}) + a_\lambda(\vec{k})\, a_\lambda^\dagger(\vec{k})\right)$$

    Looks even more like harmonic oscillators

- What are the $a_\lambda(\vec{k})$ operators?
  - Want to make analogy to ladder operators

$$[a, a^\dagger] = 1 \longrightarrow [a_\lambda(\vec{k}), a_{\lambda'}^\dagger(\vec{k}')] = \delta_{\lambda\lambda'}\delta_{\vec{k}\vec{k}'}$$

+ If we demand that,

$$H = \sum_{\vec{k}} \hbar\omega_{\vec{k}} \sum_{\lambda} \left(a_\lambda^\dagger(\vec{k})\, a_\lambda(\vec{k}) + \tfrac{1}{2}\right)$$ ← divergent sum

The infinite term is the <u>zero-point</u> or <u>vacuum energy</u>.

It is usually dropped but relates to the <u>cosmological constant</u> + <u>Casimir Effect</u>

+ To construct states, start with vacuum $|0\rangle$ with $a_\lambda(\vec{k})|0\rangle = 0$

Then <u>single-photon</u> states are $a_\lambda^\dagger(\vec{k})|0\rangle \equiv |\vec{k},\lambda\rangle$

Double-photon states are $a_\lambda^\dagger(\vec{k})\, a_{\lambda'}^\dagger(\vec{k}')|0\rangle = |\vec{k},\lambda;\vec{k}',\lambda'\rangle$, etc.

This basis (or type) of Hilbert space is called <u>Fock space</u>

A general state is $|n_{\lambda_1}(\vec{k}_1); n_{\lambda_2}(\vec{k}_2); \ldots\rangle = \frac{(a_{\lambda_1}^\dagger(\vec{k}_1))^{n_1}}{\sqrt{n_1!}} \cdots |0\rangle$

+ Based on the commutators, these are orthonormal, like harmonic oscillators.

Similarly,

annihilation operator $\rightsquigarrow a_\lambda(\vec{k})\,|n_{\lambda_1}(\vec{k}_1);\ldots\rangle = \sqrt{n_{\lambda_1}(\vec{k}_1)}\,\delta_{\lambda\lambda_1}\delta_{\vec{k}\vec{k}_1}\,|n_{\lambda_1}(\vec{k}_1)-1;\ldots\rangle + \ldots$

creation $\rightsquigarrow a_\lambda^\dagger(\vec{k})\,|n_{\lambda_1}(\vec{k}_1);\ldots\rangle = \sqrt{n_{\lambda_1}(\vec{k}_1)+1}\,\delta_{\lambda\lambda_1}\delta_{\vec{k}\vec{k}_1}\,|n_{\lambda_1}(\vec{k}_1)+1;\ldots\rangle + \ldots$

- A few other comments

  + Photons have spin 1 but only 2 spin states given by the circular polarizations $\hat{\epsilon}_\pm(\vec{k})$. This is fundamentally related to gauge freedom.

  + Just like they have energy, photons carry momentum

$$\vec{P} = \sum_{\vec{k}} \sum_{\lambda} \hbar\vec{k}\, a_\lambda^\dagger(\vec{k})\, a_\lambda(\vec{k})$$

  + The energy of a state is the sum of individual photon energies

$$E = \sum_{\vec{k},\lambda} \hbar\omega_{\vec{k}}\, n_\lambda(\vec{k})$$

  + Photons are bosons (spin 1), and the states are symmetric under exchange of photons b/c $a_\lambda^\dagger(\vec{k})$ operators commute.