- Angular Momentum 

We will start by building an algebraic framework to understand the angular equation

— Angular Momentum Operators

We've so far only talked about position + momentum operators

- Angular momentum is another possibility $\vec{L} = \vec{X} \times \vec{p}$

$$L_x = yp_z - zp_y, \; L_y = zp_x - xp_z, \; L_z = xp_y - yp_x$$

+ The key is the commutator

$$[L_x, L_y] = [yp_z, zp_x] + [zp_y, xp_z] = yp_x[p_z, z] + xp_y[z, p_z]$$
$$= i\hbar(xp_y - yp_x) = i\hbar L_z$$

+ All together,

$$[L_x, L_y] = i\hbar L_z, \; [L_y, L_z] = i\hbar L_x, \; [L_z, L_x] = i\hbar L_y$$

a $[L_i, L_j] = i\hbar \sum_k \epsilon_{ijk} L_k$ ← define angular momentum

+ Can define "raising" and "lowering" operators

$$L_\pm = L_x \pm iL_y, \; [L_z, L_\pm] = i\hbar L_y \pm i(-i\hbar L_x) = \pm\hbar L_\pm$$

If $L_z|\mu\rangle = \mu|\mu\rangle$, then $L_z(L_\pm|\mu\rangle) = L_\pm L_z|\mu\rangle \pm \hbar L_\pm|\mu\rangle = (\mu \pm \hbar)L_\pm|\mu\rangle$

- The differential operators follow from

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$ (Notice that denominators are length elements)

+ or $L_z = -i\hbar\frac{\partial}{\partial\phi}$, $L_\pm = \pm\hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\phi}\right)$

+ Apparently the wavefunction for eigenstate $L_z|\mu\rangle = \mu|\mu\rangle$ is $\langle\phi|\mu\rangle \propto e^{i\mu\phi/\hbar}$. To be single valued, $\mu = m\hbar$, $m = 0, \pm1, \pm2, \ldots$

- We also define a total angular momentum operator

$$\vec{L}^2 = L_x^2 + L_y^2 + L_z^2$$

+ The commutator

$$[\vec{L}^2, L_x] = [L_y^2, L_x] + [L_z^2, L_x] = L_y[L_y, L_x] + [L_y, L_x]L_y + L_z[L_z, L_x] + [L_z, L_x]L_z$$

$$= -i\hbar L_y L_z - i\hbar L_z L_y + i\hbar L_z L_y + i\hbar L_y L_z = 0$$

That is $\vec{L}^2$ commutes with $L_x, L_y, L_z$ $\quad [\vec{L}^2, L_i] = 0$

+ Even though there are not simultaneous eigenstates of $L_x, L_y, L_z$, there are simultaneous eigenstates of $\vec{L}^2$ and one of $L_i$. Choose $L_z$.

$$L_z|\lambda,\mu\rangle = \mu|\lambda,\mu\rangle \quad , \quad \vec{L}^2|\lambda,\mu\rangle = \lambda|\lambda,\mu\rangle$$

+ The differential operator is

$$L^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$ (exercise for reader)

If we compare back to Schrödinger eqn, the angular part is

$$L^2 Y(\theta,\phi) = +\ell(\ell+1)\hbar^2 Y(\theta,\phi)$$

- Spherical Harmonics

These are the eigenfunctions $Y^{m}_{\lambda\mu}(\theta,\phi) = \langle\theta,\phi|\lambda,\mu\rangle$

- Start by remembering $\mu = m\hbar$ for $m$ integer, $Y^m_\lambda(\theta,\phi) \propto e^{im\phi}$

+ Then

$$L^2 Y^m_\lambda = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y^m_\lambda}{\partial\theta}\right)\right] + \frac{m^2\hbar^2}{\sin^2\theta}Y^m_\lambda = \lambda Y^m_\lambda$$

+ Rewrite $x = \cos\theta$ Then

$$(1-x^2)\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial Y}{\partial x}\right] + \left[\frac{\lambda}{\hbar^2}(1-x^2) - m^2\right]Y = 0$$

This is the associated Legendre equation

+ Note that we have written (by comparison) $\lambda = \ell(\ell+1)\hbar^2$

Solutions to assoc. Legendre eqn are singular unless $\ell = 0, 1, 2, \ldots$ and $|m| \le \ell$. That is, $m = -\ell, -\ell+1, \ldots, \ell-1, \ell$.

Therefore, we write eigenstates as $|\ell, m\rangle$ and eigenfunctions as $\langle\theta,\phi|\ell,m\rangle = Y^m_\ell(\theta,\phi) \leftarrow$ spherical harmonics

- Nonsingular solutions:
  Up to normalization, $Y_\ell^m(\theta,\phi) \propto P_\ell^m(\cos\theta)\, e^{im\phi}$
  + For $m=0$, we have $P_\ell^0(x) = P_\ell(x)$, the Legendre polynomials
  See book for definition. Some examples:
  $$P_0(x) = 1 \;,\; P_1(x) = x \;,\; P_2(x) = \tfrac{1}{2}(3x^2-1)\;, \ldots$$
  + Associated Legendre functions: $P_\ell^m(x) = (1-x^2)^{|m|/2}\left(\frac{d}{dx}\right)^{|m|} P_\ell(x)$
  Note that these vanish for $|m| > \ell$
  + Examples (also given in text)
  $$P_0^0(\cos\theta) = 1 \;,\; P_1^{\pm1}(\cos\theta) = \sin\theta,\; P_1^0(\cos\theta) = \cos\theta$$
  $$P_2^0(\cos\theta) = \tfrac{1}{2}(3\cos^2\theta - 1),\; P_2^{\pm1}(\cos\theta) = 3\sin\theta\cos\theta,\; P_2^{\pm2}(\cos\theta) = 3\sin^2\theta$$
  Notice that odd $m$ has an odd power of $\sin\theta$.
- The normalized spherical harmonics are
  $$Y_\ell^m(\theta,\phi) = (-1)^m \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\; e^{im\phi}\, P_\ell^m(\cos\theta) \quad,\quad m \geq 0$$
  $$Y_\ell^{-m}(\theta,\phi) = (-1)^m \left(Y_\ell^m(\theta,\phi)\right)^*$$
  + The signs sometimes are given with different conventions
  + Since $Y_\ell^m(\theta,\phi)$ are eigenfunctions of $\vec{L}^2$ and $L_z$ with different eigenvalues, they are orthogonal (and now normalized)
  $$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \sin\theta\, (Y_{\ell'}^{m'})^* Y_\ell^m = \delta_{\ell,\ell'}\,\delta_{m,m'}$$
  + From our operator relations
  $$\vec{L}^2 \cdot Y_\ell^m = \ell(\ell+1)\hbar^2 Y_\ell^m \;,\; L_z \cdot Y_\ell^m = m\hbar Y_\ell^m \;,\; L_\pm \cdot Y_\ell^m \propto Y_\ell^{m\pm1}$$
  + $\ell$ is the azimuthal quantum number or total orbital angular momentum
  $m$ is the magnetic quantum number or orbital $z$ angular momentum