PHYS-3301 Winter Homework 9 Due 15 Mar 2017

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Derivatives Have Lowered Indices**

   As discussed in the class notes, 4-vectors with raised or lowered indices have the following Lorentz transformations:

   $$a^{\mu'} = \Lambda^{\mu'}{}_{\nu} a^{\nu} \quad \text{and} \quad a_{\mu'} = \Lambda^{\nu}{}_{\mu'} a_{\nu} \ , \tag{1}$$

   where $[\Lambda^{\mu'}{}_{\nu}]$ is the usual Lorentz transformation matrix from $S \to S'$ and $[\Lambda^{\nu}{}_{\mu'}]$ is its matrix inverse (the transformation from $S' \to S$).

   (a) Using the fact that the spacetime position x^{μ} is a 4-vector, find the partial derivatives $\partial x^{\mu}/\partial x^{\nu'}$ and $\partial x^{\mu'}/\partial x^{\nu}$ in terms of $\Lambda^{\mu'}{}_{\nu}$ and $\Lambda^{\nu}{}_{\mu'}$. *Hint:* For two positions as measured in the same frame, $\partial x^{\mu}/\partial x^{\nu} = \delta^{\mu}_{\nu}$ (think about why).

   (b) If f is a Lorentz invariant function (meaning its value at a fixed spacetime point is the same in any frame — like the temperature), use the multivariable chain rule to show that

   $$\frac{\partial f}{\partial x^{\mu'}} = \Lambda^{\nu}{}_{\mu'} \frac{\partial f}{\partial x^{\nu}} \ . \tag{2}$$

   In other words, you are showing that a partial derivative has the same transformation as a 4-vector with a lowered index. As a result, people will usually write $\partial_{\mu} f \equiv \partial f/\partial x^{\mu}$.

2. **Boosts and Rotations**

   In matrix form, we can define the boost Λ_{tx} along x and the rotation Λ_{xy} in the xy plane (around the z axis) as follows:

   $$\Lambda_{tx}(\phi) = \begin{bmatrix} \cosh\phi & -\sinh\phi & & \\ -\sinh\phi & \cosh\phi & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \ , \quad \Lambda_{xy}(\theta) = \begin{bmatrix} 1 & & & \\ & \cos\theta & \sin\theta & \\ & -\sin\theta & \cos\theta & \\ & & & 1 \end{bmatrix} \ . \tag{3}$$

   Empty elements in the matrices above are zero.

   (a) In matrix form, the metric $\eta_{\mu\nu}$ is

   $$\eta = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \ . \tag{4}$$

   Show that both rotation and boost in (3) satisfy the condition $\eta_{\mu\nu} = \Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}{}_{\nu}\eta_{\alpha\beta}$, which is $\eta = \Lambda^T \eta \Lambda$ in matrix notation.

   (b) Consider two successive boosts along x, $\Lambda_{tx}(\phi_1)$ and $\Lambda_{tx}(\phi_2)$. Show that these multiply to give a third boost $\Lambda_{tx}(\phi_3)$ and find ϕ_3. Using the relationship $v/c = \tanh\phi$ between velocity and rapidity ϕ, reproduce the velocity addition rule. *Hint:* You will need the angle-addition rules for hyperbolic trig functions.

(c) First, write down the Lorentz transformation matrix $\Lambda_{ty}(\phi)$ corresponding to a boost along the y direction by permuting axes. Then show that you can get a boost along y by rotating axes, boosting along x, then rotating back by proving that $\Lambda_{ty}(\phi) = \Lambda_{xy}(-\pi/2)\Lambda_{tx}(\phi)\Lambda_{xy}(\pi/2)$.

3. **Some Scalar Products**

In some frame, the components of two 4-vectors are

$$a^\mu = (2, 0, 0, 1) \text{ and } b^\mu = (5, 4, 3, 0) \ . \tag{5}$$

inspired by a problem in Hartle

(a) Find a^2, b^2, and $a \cdot b$.

(b) Does there exist another inertial frame in which the components of a^μ are $(1, 0, 0, 1)$? What about b^μ? Explain your reasoning.

Now consider lightlike 4-vectors f^μ and g^μ.

(c) If f^μ and g^μ are orthogonal ($f \cdot g = 0$), prove that they are parallel ($f^\mu \propto g^\mu$).

(d) Is the 4-vector $f^\mu + g^\mu$ spacelike, timelike, or lightlike? Assume that both $f^0 > 0$ and $g^0 > 0$.