PHYS-3301 Winter Homework 4 Due 1 Feb 2017

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Rotations and Orthogonal Matrices**

   Two position vectors have the dot product $\vec{x} \cdot \vec{y} = x^i y^i$ (remember, superscripts tell you the component, and we use Einstein summation convention). The length of a vector $|\vec{x}|$ is given by the vector's dot product with itself as follows: $|\vec{x}|^2 = \vec{x} \cdot \vec{x}$.

   (a) Rotations leave the lengths of vectors invariant (unchanged). Prove that this means that dot products are invariant under rotations.

   (b) Use the invariance of all dot products under rotations to show that all rotation matrices R satisfy the identity
   $$R^i{}_j R^i{}_k = \delta_{jk} \ , \tag{1}$$
   where δ_{jk} is the Kronecker delta symbol (equal to 1 if $j = k$ and 0 otherwise). *Hint:* To prove (1), consider frame S' rotated by R with respect to frame S. Then write the dot product $\vec{x}' \cdot \vec{y}'$ in terms of $\vec{x}, \vec{y}$ and set it equal to $\vec{x} \cdot \vec{y}$. You get an equation true for all $\vec{x}, \vec{y}$, which allows you to cancel the vector components from both sides of the equation.

   (c) Treat each column of R as a vector. Show that all the columns are perpendicular to each other and have length 1. This means that rotation matrices are *orthogonal matrices.*

2. **Choosing Frames Wisely**

   In each part, clearly state what inertial reference frame you use to solve the problem.

   (a) An airplane flies at a constant velocity toward downtown Winnipeg. At some point, the pilot drops a care package of chocolate bars, which lands in the lawn in front of Wesley Hall. In which direction should the pilot look to see the happy faces on all the chocoholic students when the package lands? Explain, and ignore air resistance.

   (b) *Barton 2.2 rephrased* A river flows at 5 km/hr, and a boat in it can move 8 km/hr relative to the water. As the boat moves upstream, the driver hears a splash but only realizes that it was the life preserver falling overboard 15 minutes later. The driver turns around and heads to retrieve the life preserver. How soon can the boat catch up to the life preserver?

   (c) *from Barton 2.3* Train 1 of length L_1 moves right along a track with speed u_1, while train 2 of length L_2 moves left along a parallel track with speed u_2. Consider event A, the front of train 1 passes the front of train 2; event B, the front of train 1 passes the rear of train 2; and event C, the front of train 2 passes the rear of train 1. What are the lengths of time between events A and B and between events A and C? (*Hint:* first argue that these lengths of time are the same in all inertial frames.)

3. **Parallel Axis Theorem**

   The *moment of inertia* of a collection of particles around a given axis is $I \equiv \sum_i m_i r_i^2$, where r_i is the distance of the individual particle from the axis of rotation (ie, if the axis of rotation is the z axis, r_i is the distance in the x, y plane). For a continuous object, replace the sum by an integral overinfinitesimal mass elements dm.

Show that the moment of inertia around an axis a perpendicular distance h away from the object's center of mass is $I = I_{CM} + Mh^2$, where M is the total mass and I_{CM} is the moment of inertia around a parallel axis running through the center of mass, as in the figure.