PHYS-3301 Winter Homework 2 Due 25 Jan 2017

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **How to Calculate Gaussian Integrals**

   (a) Start by evaluating the integral

   $$J(a) \equiv \int_0^\infty dx\, x e^{-ax^2} \ . \tag{1}$$

   Hint: it's probably easiest to use the substitution $u = x^2$.

   (b) Evaluate the integral

   $$I(a) \equiv \int_0^\infty dx\, e^{-ax^2} \tag{2}$$

   as follows:

   i. Write

   $$(I(a))^2 = \int_0^\infty dx \int_0^\infty dy\, e^{-ax^2} e^{-ay^2} \tag{3}$$

   (be careful to rename dummy integration variables) and then convert to plane polar coordinates.

   ii. You should now be able to us your result for $J(a)$ to calculate $(I(a))^2$. Then find $I(a)$. Your answer should be $I(a) = (1/2)\sqrt{\pi/a}$.

   (c) Calculate the integral

   $$\int_0^\infty dx\, x^2 e^{-ax^2} \tag{4}$$

   by showing first that it is equal to $-dI(a)/da$.

2. **The Ideal Gas**

   A classical ideal gas is composed of molecules of mass m with the number of molecules per energy level given by the Maxwell-Boltzmann distribution for distinguishable particles. As discussed in class, the number of particles in a state of energy E_k is

   $$n(E_k) = e^{(\mu - E_k)/k_B T} \text{ where } E_k = \frac{\hbar^2 k^2}{2m} \ , \tag{5}$$

   and the degeneracy at energy E_k is $d_k = (V/2\pi^2)k^2 dk$, as discussed in the class notes.

   (a) Using your results from the previous problem, show that the number of air molecules is given by $N/V = (mk_BT/2\pi\hbar^2)^{3/2} \exp(\mu/k_BT)$.

   (b) Using your results from the previous problem, show that the total energy is $3Nk_BT/2$.

3. **Cold Electron Gas**

   Electrons confined in a piece of metal can also be described as particles in a 3D infinite square well. They have 2 spin states, so their energy levels have twice the degeneracy of an ideal gas $d_k = (V/\pi^2)k^2 dk$. They follow the Fermi-Dirac distribution; as $T \to 0$, this distribution turns in to a step function that is equal to 1 for $E_k < \mu$ and 0 for $E_k > \mu$. Find the relationships between the number N of electrons and μ and between total energy E and μ.

4. **Stefan-Boltzmann Formula** *like a problem in Griffiths (and many other places)*

   Now we consider photons following the Planck law discussed in class. Integrate the Planck distribution over all frequencies to show that the energy density of light at temperature T is $E/V = \alpha(k_B T)^4/(\hbar c)^3$, where α is a numerical constant (you do not need to evaluate α but can leave it in the form of a dimensionless integral). This is one form of the Stefan-Boltzmann formula, which was actually known before the Planck law.