

② Momentum + Angular Momentum Wavefunctions

— Remember that only eigenfunctions of the momentum operator describe particles with a definite momentum

• Momentum operator is a derivative $\vec{p} = -i\hbar \vec{\nabla}$ or $p = -i\hbar \frac{d}{dx}$

• An eigenfunction $\psi_{\vec{p}}$ satisfies $\vec{p} \cdot \psi_{\vec{p}} \stackrel{\text{operator action}}{=} -i\hbar \vec{\nabla} \psi_{\vec{p}}(\vec{x}) \stackrel{\text{multiplication}}{=} \vec{p} \psi_{\vec{p}}(\vec{x})$

+ In 1D, this is $-i\hbar \frac{d\psi_p(x)}{dx} = p \psi_p(x) \Rightarrow \psi_p(x) \propto e^{ipx/\hbar}$

+ In 3D, it's just a product: $\psi_{\vec{p}}(\vec{x}) \propto e^{ip_x x/\hbar} e^{ip_y y/\hbar} e^{ip_z z/\hbar} = e^{i\vec{p} \cdot \vec{x}/\hbar}$

• + The eigenvalue p (or \vec{p}) is the definite momentum

• The allowed values of momentum depend on boundary conditions

+ For periodic b.c., $\psi(x) = \psi(x + 2\pi R)$, we need $p = \hbar n/R/L$ for n integers. Then

$$\psi_p(x) = \frac{1}{\sqrt{2\pi R}} e^{inx/R} \text{ (normalized)}$$

+ For Dirichlet or Neumann b.c., momentum eigenfunctions are not allowed!

+ For the whole real line $-\infty < x < \infty$, any p is allowed, but ψ_p is not normalizable! We still write $\psi_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$

— But you can actually write any wavefunction in terms of momentum eigenfunctions

• Remember the Fourier series for any periodic function:

$$\begin{aligned} \text{+ If } \psi(x) = \psi(x + 2\pi R), \quad \psi(x) &= \sum_{n=-\infty}^{\infty} \frac{c_n}{\sqrt{2\pi R}} e^{inx/R} = \sum_n c_n \psi_{p=\hbar n/R}(x) \\ &= \sum_n c_n \psi_{p=\hbar n/R}(x) \end{aligned}$$

+ The Fourier coefficient is $c_n = \frac{1}{\sqrt{2\pi R}} \int_0^{2\pi R} dx e^{-inx/R} \psi(x)$
 $= \int dx \psi_p^*(x) \psi(x)$

+ These coefficients hold all the same information as the wavefunction! It all relates to inner products + linear algebra

• On the full real line, we have the Fourier transform

$$+ \psi(x) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \phi(k) e^{ikx} = \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi\hbar}} \tilde{\psi}(p) \psi_p(x)$$

where $p = \hbar k$ and $\tilde{\psi}(p = \hbar k) = \phi(k) / \sqrt{\hbar}$

+ We call $\phi(k)$ the Fourier transform and $\tilde{\psi}(p)$ the momentum space wavefunction

+ The inverse transform gives $\tilde{\psi}(p) = \int_{-\infty}^{\infty} dx \psi_p^*(x) \psi(x)$

+ Doing the F.T. followed by inverse F.T. shows

$$\int \frac{dk}{2\pi} e^{ik(x-x')} = \delta(x-x')$$

See reading + HW for more on the Dirac delta function

• Properties of the momentum space wavefunction

+ Just like $x \cdot \psi(x) = x \psi(x)$ (ie, operator action = multiplication)

$$p \cdot \tilde{\psi}(p) = p \tilde{\psi}(p)$$

+ But conversely, $x \cdot \tilde{\psi}(p) = i\hbar \frac{d\tilde{\psi}(p)}{dp}$ (you will prove)

+ If $\psi(x)$ is normalized, so is $\tilde{\psi}(p)$: $\int_{-\infty}^{\infty} dp |\tilde{\psi}(p)|^2 = 1$

+ This is just as "valid" of a wavefunction as the usual one.

- Angular Momentum Eigenfunctions

• Similarly, particles with definite angular momentum have wavefunctions that are eigenfunctions of \vec{L} operators

+ Actually, we can only have eigenfunctions of L_z and \vec{L}^2 simultaneously, not L_x , L_y , and L_z

+ The eigenfunctions of L_z + \vec{L}^2 are called spherical harmonics Y_l^m

+ $L_z = -i\hbar \frac{\partial}{\partial \phi}$ and $\vec{L}^2 = -\hbar^2 \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right]$
(angular part of Laplacian)

+ This means $Y_l^m(\theta, \phi) = f(\theta) e^{im\phi}$

Since ϕ has periodicity 2π , $m = \text{integer}$, $L_z \cdot Y_l^m = m\hbar Y_l^m$

+ The equation $\vec{L}^2 \cdot Y_l^m = \hbar^2 l(l+1) Y_l^m$ is actually known as a D.E. $f(\theta)$ is a special type of function called an "associated Legendre function." Non-singular only for $l=0, 1, \dots$, $m = -l, -l+1, \dots, l$

• Any function of angles on a sphere can be written as a sum of spherical harmonics (like Fourier series)

$$+ g(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l a_{lm} Y_l^m(\theta, \phi)$$

+ Again, a_{lm} coefficients contain all the information

+ The coefficients are

$$a_{lm} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin \theta (Y_l^m(\theta, \phi))^* g(\theta, \phi)$$

+ This is used in quantum mechanics, electromagnetic theory, astronomy, ...

$$+ \text{You can also see } \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin \theta (Y_{l'}^{m'})^* Y_l^m(\theta, \phi) = \delta_{l',l} \delta_{m',m}$$