PHYS-2106 Winter Homework 9 Due 20 Mar 2017

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

Hint: You will want to remind yourself about integration by parts and other integration tricks (like differentiation with respect to parameters of the integral) for this assignment.

1. **Some Fourier Transforms**

   Find the Fourier transforms $\tilde{f}(k)$ for each of the following functions $f(x)$. $\Theta(x)$ is the Heaviside step function.

   (a) $f(x) = x\Theta(a - |x|)$

   (b) $f(x) = (a - |x|)\Theta(a - |x|)$

   (c) $f(x) = x\exp(-x^2/2\sigma^2)$ *Hint:* See the example in RHB equation (13.7) and use properties of the Fourier transform.

2. **Integrals from the Inverse Transform and Parseval's Theorem** *inspired by Boas & Arfken-Weber*

   (a) First, find the Fourier transform $\tilde{f}(k)$ of the function $f(x) = \exp(-\alpha|x|)$.

   (b) Now, use the inverse transform $f(x) = \mathcal{F}^{-1}[\tilde{f}](x)$ to show that

   $$\int_0^\infty dk\, \frac{\cos(kx)}{k^2 + \alpha^2} = \frac{\pi}{2\alpha} e^{-\alpha|x|} \ . \tag{1}$$

   (c) Finally, use Parseval's theorem for the Fourier transform to evaluate $\int_{-\infty}^{\infty} dk\, 1/(\alpha^2 + k^2)^2$.

3. **Impedance**

   In an alternating-current circuit, the current and potential obey a generalized form of Ohm's law $\tilde{V}(\omega) = \tilde{I}(\omega)Z(\omega)$ for each frequency, where $Z(\omega)$ is known as the *impedance.* Since this multiplicative relationship is valid for the variables as a function of frequency and *not* time, it is really a relation between the Fourier transforms of potential and current (time t and frequency ω are Fourier conjugate variables, like x and k).

   The impedance of a circuit with resistance R, inductance L, and capacitance C is $Z(\omega) = R + i(L\omega - 1/C\omega)$.

   (a) If the capacitance $C \to \infty$, show that $V(t) = RI(t) + L\dot{I}(t)$, where the dot is a time derivative.

   (b) If $R = 0$ and $L = 0$, show that $I(t) = C\dot{V}(t)$.

4. **Convolution**

   As we discussed in class, the picture $g(x)$ of some image function taken by a measuring device is the convolution of the device's transfer function $K(x)$ with the true image $f(x)$:

   $$g(x) = \int_{-\infty}^{\infty} dy\, f(y)K(x - y) \ . \tag{2}$$

   In this problem, suppose that the transfer function is $K(x) = \exp(-\alpha|x|)$.

(a) Suppose that the true image is a single point $f(x) = \delta(x)$. What is the picture function taken by the device?

(b) Using your result from question 2(a) and the convolution theorem, show that the true image $f(x)$ that yields the picture $g(x)$ is $f(x) = \alpha g(x)/2 - g''(x)/2\alpha$, where a prime is a derivative with respect to x.