PHYS-2106 Winter Homework 7 Due 6 Mar 2017

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

Hint: You will want to remind yourself about integration by parts and trigonometric identities (like angle addition formulas, etc) for this assignment.

1. **Evaluating Fourier Coefficients** *from a variety of sources*

   The following functions are defined as follows for the range $-\pi < x < \pi$. Assuming they have period 2π, find the Fourier series (in terms of real trigonometric functions) for each function.

   (a) $f(x) = x$ (this is known as a *sawtooth wave*)

   (b) $f(x) = x^2/2$ (*Hint:* you can integrate your previous result and then find the constant term.)

   (c) $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ \sin(x) & 0 \le x < \pi \end{cases}$ (this is the output of a *half-wave rectifier*)

   (d) $f(x) = \cosh(x/\pi)$ (remember that the hyperbolic cosine is $\cosh\theta = (e^\theta + e^{-\theta})/2$)

2. **Fourier Series and Sums**

   By evaluating Fourier series at a specific value of the argument, we can carry out infinite sums. In this problem, define the function $f(x)$ which is periodic with period 2 defined such that $f(x) = |x|$ for $-1 < x < 1$.

   (a) Show that the Fourier coefficients for $f(x)$ are $a_0 = 1$, $a_n = -4/\pi^2 n^2$ for n odd, $a_n = 0$ for $n > 0$ even, and $b_n = 0$.

   (b) Show that $\sum_{k=0}^{\infty} 1/(2k+1)^2 = \pi^2/8$. *Hint:* use the Fourier series for $f(1)$ and simplify.

   (c) Show that $\sum_{k=0}^{\infty} (-1)^k/(2k+1) = \pi/4$. *Hint:* Differentiate $f(x)$ and its Fourier series and evaluate them for $x = 1/2$.

   (d) *almost RHB 12.16* For something a little different, use your result from question 1(d) to show that $\sum_{n=1}^{\infty} 1/(n^2\pi^2 + 1) = 1/(e^2 - 1)$.

3. **Successive Approximations**

   In this problem, we will use Maple software to see how using more and more terms in the Fourier series approximates a function better and better. We consider the function $f(x)$ which is periodic with period 2 defined such that $f(x) = |x|$ for $-1 < x < 1$ and use the Fourier series defined in problem 2(a). Attach a printout of your Maple worksheet at the end of your assignment.

   (a) In a Maple worksheet, define $f(x)$ and a function $g(x)$ which is given by the first two nonzero terms of the Fourier series for $f(x)$. Then plot both of these functions on the same graph for $-1 < x < 1$; show $f(x)$ as a solid line and $g(x)$ as a dashed line. You can use the following example Maple code if you wish.

   ```
   f:=x->|x|
   g:=x->(1/2) -(4/Pi^2)*cos(Pi*x)
   plot([f(x),g(x)],x=-1..1,linestyle=[solid,dash])
   ```

   Note: these are the keystrokes used to enter the code, but it will look different in Maple.

(b) Next, define a function $h(x)$ which is given by the first four nonzero terms of the Fourier series for $f(x)$. Then plot $f(x)$ and $h(x)$ on the same plot. You can copy and modify some of your previous code.

(c) Finally, plot the differences $f(x) - g(x)$ and $f(x) - h(x)$ on the same plot. Describe how the error in the approximation changes as we increase the number of terms in the approximate Fourier series. At what points is the absolute error largest in each approximation?