PHYS-2106 Winter Homework 6 Due 27 Feb 2017

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Gaussian Probabilities**

   A randomly distributed Gaussian variable x with mean value μ and standard deviation (the spread of values) σ has probability distribution

   $$P(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] . \quad (1)$$

   That means that the probability for x to be found between a and b is $\int_a^b dx P(x)$.

   (a) Write the probability that x is found in the range $-\infty < x < b$ (this is the *cumulative probability*) in terms of the error function as defined in class/the text. Using either table 30.3 in Riley-Hobson-Bence or the `erf` and `evalf` commands in Maple, find the cumulative probability that $x < \mu + 3\sigma$. Give your answer to 4 significant figures. If you use Maple, attach your code to your homework.

   (b) In terms of the error function, find the total probability that x lies within a distance $n\sigma$ of its mean value $\mu - n\sigma < x < \mu + n\sigma$.

   (c) Experimental particle physicists do not announce a discovery unless the experimental results are at least 5σ away from the "no discovery" value. Assuming a Gaussian probability distribution (1) for experimental statistical errors, use the `erf` and `evalf` commands in Maple to find the probability that $|x - \mu| > 5\sigma$. Give your answer to 1 significant figure.

2. **Maxwell Distribution**

   In an ideal gas, the number of gas molecules per volume with speed between v and $v + dv$ is

   $$4\pi n \left(\frac{m}{2\pi k_B T}\right)^{3/2} v^2 e^{-mv^2/2k_BT} dv , \quad (2)$$

   where n is the total number density of gas molecules (number per volume), $\hbar$ is Planck's constant, m is the mass of a single gas molecule (assuming only one type), k_B is Boltzmann's constant, and T is the gas temperature. This is the Maxwell distribution.

   Show that the total energy density (kinetic energy per volume) of the gas is $3nk_BT/2$. Remember that the kinetic energy of each gas molecule is $mv^2/2$.

3. **Some Integrals** *partly from Spiegel*

   Evaluate the following integrals. You may leave your answer in the form $\Gamma(x)$ unless x is a positive integer or a half-integer. In those cases, give the explicit value (which could include powers of π).

   (a)

   $$\int_0^\infty dx\, x^{1/4} e^{-\sqrt{x}} \quad (3)$$

   (b)

   $$\int_0^\infty dx\, x^6 e^{-3x} \quad (4)$$

(c)

$$\int_{-\infty}^{\infty} dx\, e^{5x} \sin(4x)\delta(x) \tag{5}$$

(d)

$$\int_{-\infty}^{\infty} dx\, e^{5x} \sin(4x)\delta'(x) \tag{6}$$

Remember that a prime indicates a derivative with respect to x.

4. **Hat Function**

   Define a "hat function" $f(x)$ such that

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & 0 < x < 1 \\ 0 & x > 1 \end{cases} , \tag{7}$$

   with $f(0)$ and $f(1)$ unspecified. Find two different ways to write $f(x)$ in terms of Heaviside step functions and give the values of $f(0)$ and $f(1)$ for each formula.

5. **3D Delta Function and the Laplacian**

   This problem will lead you through a proof of the statement that

$$\nabla^2(1/r) = -4\pi\delta^3(\vec{x}) , \tag{8}$$

   where ∇^2 is the 3D Laplacian, r is the radial coordinate in spherical coordinates, and $\delta^3(\vec{x})$ is the 3D delta function centered at the origin. Remember that $\delta^3(\vec{x})$ is defined as zero everywhere except the origin but integrates to 1 when the region of integration includes the origin. This is a fundamental relationship in electromagnetism, and some of the calculations should remind you of electrostatic potentials and Gauss's law.

   (a) Start by using the Laplacian in spherical polar coordinates (as given in RHB table 10.3 or the equation at the bottom of page 363) to show that $\nabla^2(1/r) = 0$ for all points other than the origin. Note that this calculation is not valid at the origin because of division by zero there.

   (b) Next, remember that $\nabla^2\phi(\vec{x}) = \vec{\nabla} \cdot (\vec{\nabla}\phi)$ for any function ϕ. Define $\vec{a} = \vec{\nabla}(1/r)$ and find $\vec{a}$. Use the gradient in spherical coordinates as given in RHB table 10.3.

   (c) Now use the divergence theorem to evaluate

$$\int_B d^3x\, \nabla^2(1/r) = \int_B d^3x\, \vec{\nabla} \cdot \vec{a} = \oint_S d\vec{S} \cdot \vec{a} , \tag{9}$$

   where B is interior of a sphere of radius R centered at the origin and S is the surface of that sphere. You should find that the surface integral gives -4π, which proves (8).