PHYS-2106 Winter Homework 5 Due 13 Feb 2017

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Exponential of a Hermitian Matrix**

   Let A be a Hermitian matrix and consider the matrix $U = \exp[-iA]$ defined by the Taylor expansion of the exponential.

   (a) Show that the eigenvectors of A are eigenvectors of U. If the eigenvalues of A are a_i for $i = 1, \cdots N$, show that the eigenvalues of U are $\exp[-ia_i]$.

   (b) Show that U is unitary.

   This example is important in quantum mechanics when $A = Ht/\hbar$, where H is the Hamiltonian operator, t is time, and $\hbar$ is Planck's constant. Then U evolves the wavefunction over a time t.

2. **Unitary Change of Basis** *extended from RHB 8.21*

   Define the Hermitian matrix
   $$B = \begin{bmatrix} 10 & 3i \\ -3i & 2 \end{bmatrix} . \tag{1}$$

   (a) Find the eigenvalues and normalized eigenvectors of B.

   (b) Find the unitary matrix U such that $U^\dagger BU = B'$ is diagonal and give the diagonal elements of B'. Verify that your matrix U is unitary.

   (c) Write the vector $y = \begin{bmatrix} 2 & 5 \end{bmatrix}^T$ as a linear combination of the eigenvectors of B. In other words, find the components of y in the basis of eigenvectors of B.

3. **More About Eigenvector Basis Sets** *inspired by RHB 8.17 and Arfken & Weber*

   (a) Define the matrices
   $$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} , \quad B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} , \quad C = \begin{bmatrix} 5 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix} . \tag{2}$$

   Without finding the eigenvalues or eigenvectors, show that A and B have a simultaneous basis of eigenvectors.

   (b) By matrix multiplication, show that the vectors $x = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T$, $y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$, and $z = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}^T$ are eigenvectors of both A and B and find their eigenvalues for each matrix.

   (c) Find an orthonormal basis of eigenvectors for C. *Hint:* If you choose carefully, you can find the eigenvectors to be orthogonal from the start. Otherwise, you may need to use the Gram-Schmidt process to orthonormalize your initial choice.

4. **Short Proofs**

   Prove the following statements (your answers should be short):

   (a) The diagonal elements of an antisymmetric matrix are all zero.

(b) The trace of a diagonalizable matrix is the sum of its eigenvalues. *Hint:* Relate the matrix to its diagonalizable form by a similarity transformation.

(c) The determinant of a diagonalizable matrix is the product of its eigenvalues.