PHYS-2106 Winter Homework 3 Due 30 Jan 2017

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Matrix Powers**

   (a) On the last homework, we defined the Pauli matrix

   $$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} . \tag{1}$$

   Show that $\exp[i\theta\sigma_x] = \cos\theta + i\sigma_x \sin\theta$, where θ is any angle.

   (b) *from Arfken & Weber* Define the matrix

   $$K = \begin{bmatrix} 0 & 0 & i \\ -i & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} . \tag{2}$$

   Find the smallest positive power n of K such that $K^n = 1$.

2. **Evaluating Determinants** *extended from RHB 8.2*

   Evaluate the following determinants. Simplify as much as possible.

   (a) $\det \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

   (b) $\det \begin{bmatrix} gc & ge & a+ge & gb+ge \\ 0 & b & b & b \\ c & e & e & b+e \\ a & b & b+f & b+d \end{bmatrix}$

   (c) Use Maple to calculate $\det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 & 3 \\ 2 & 0 & 1 & -2 & 1 \\ 3 & 3 & -3 & 4 & -2 \\ 4 & -2 & 1 & -2 & 1 \end{bmatrix}$. Attach a printout of your Maple worksheet (you may include all Maple code together at the end of your assignment). *Instructions:* First, use the `With(LinearAlgebra)` command to load the linear algebra package. Then use the `Matrix` and `Determinant` commands. For example, the determinant in part (a) would be calculated with the Maple code

   ```
   With(LinearAlgebra):
   A:=Matrix([[a,h,g],[h,b,f],[g,f,c]])
   Determinant(A)
   ```

   You can find Maple help at `http://www.maplesoft.com/support/help/`. The `Matrix` and `Determinant` commands can be found in the left-hand menu under Mathematics, then Linear Algebra, then LinearAlgebra package.

3. **Solving Linear Equations with Maple** *adapted from Arfken & Weber*

   Solve the following system of linear equations by writing them as a matrix equation and using the matrix inverse. Use Maple software to do your calculations; you may use the `Matrix`, `MatrixInverse`, and `Multiply` commands of the LinearAlegebra package. To save typing, you can define a column vector $[a\ b\ c]^T$ easily using `<a,b,c>` and matrix multiplication AB using `A.B` (note the period). Attach your Maple code to your assignment.

$$
\begin{aligned}
1.0x_1 + 0.9x_2 + 0.8x_3 + 0.4x_4 + 0.1x_5 &= 1.0 \\
0.9x_1 + 1.0x_2 + 0.8x_3 + 0.5x_4 + 0.2x_5 + 0.1x_6 &= 0.9 \\
0.8x_1 + 0.8x_2 + 1.0x_3 + 0.7x_4 + 0.4x_5 + 0.2x_6 &= 0.8 \\
0.4x_1 + 0.5x_2 + 0.7x_3 + 1.0x_4 + 0.6x_5 + 0.3x_6 &= 0.7 \\
0.1x_1 + 0.2x_2 + 0.4x_3 + 0.6x_4 + 1.0x_5 + 0.5x_6 &= 0.6 \\
0.6x_1 + 0.1x_2 + 0.2x_3 + 0.3x_4 + 0.5x_5 + 1.0x_6 &= 0.5 \qquad (3)
\end{aligned}
$$

4. **Matrix Inverses**

   (a) *from Spiegel* Given

$$
A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \qquad (4)
$$

   find A^{-1} and B^{-1}. For your information, the matrix B can be used to represent rotation of axes in the xy plane.

   (b) *from Arfken & Weber* Represent any complex number $z = a + ib$ with a, b both real as a matrix

$$
Z = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}. \qquad (5)
$$

   If Z_1 and Z_2 are the matrix representations of z_1 and z_2, show that the matrix product $Z_1 Z_2$ is the representation of $z_1 z_2$ (so the matrices obey the complex multiplication rule). Then show that the matrix Z^{-1} represents $1/z$ (for $z \neq 0$).