PHYS-2106 Winter Homework 1 Due 23 Jan 2017

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Some Matrix Arithmetic** *sampled from Keyszig and Spiegel*

   Consider the matrices

   $$A = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} , \; B = \begin{bmatrix} 0 & -2 \\ 4 & 5 \end{bmatrix} , \; C = \begin{bmatrix} 3 & 1 & 2 \\ -2 & 4 & -2 \end{bmatrix} , \; D = \begin{bmatrix} -1 \\ 5 \end{bmatrix} . \qquad (1)$$

   Calculate the following (you may just give your answers):

   (a) $A + B$

   (b) $A + B^T$

   (c) AB

   (d) BC

   (e) $D^T A$

   (f) $C^T D$

   (g) $AB - BA$ (this is sometimes written as $[A, B]$ and called the commutator)

   If you need more practice, there are more problems in the Spiegel text (and the texts on reserve). Make sure you are comfortable with matrix algebra.

2. **Transpose and Adjoint**

   For any two $M \times N$ matrices A and B and scalar λ, show that $(A + B)^T = A^T + B^T$, $(A + B)^\dagger = A^\dagger + B^\dagger$, $(\lambda A)^T = \lambda A^T$, and $(\lambda A)^\dagger = \lambda^* A^\dagger$.

3. **Pauli Matrices** *related to RHB 8.10*

   The following matrices are known as the *Pauli matrices* and are very important in quantum mechanics:

   $$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} , \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} . \qquad (2)$$

   (a) For a and b complex numbers, find $(a\sigma_x + b\sigma_y)^\dagger$.

   (b) Show that $\sigma^2 = \sigma\sigma = 1_2$ (the 2×2 identity matrix), where σ is any of the Pauli matrices.

   (c) Show that $\sigma_x\sigma_y = -\sigma_y\sigma_x = i\sigma_z$. Similar results are true if you permute the indices x, y, z.

   (d) In some basis, the inner product on a 2D vector space is given by the metric $G = 3 \cdot 1_2 + \sigma_y$. Evaluate the inner product $\langle a|b\rangle$ of the two column vectors (in this basis)

   $$|a\rangle = \begin{bmatrix} 1 \\ i \end{bmatrix} \text{ and } |b\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} . \qquad (3)$$

4. **A Linear Differential Operator, Eigenvectors, and Eigenvalues**

   Consider the space of real functions $f(x)$ of a variable x as a vector space. The second derivative D is a linear operator on this space. Specifically, if $|f\rangle = f(x)$, $D|f\rangle = d^2f/dx^2$. The eigenvectors of this operator (also known as eigenfunctions) are therefore functions that satisfy the equation $d^2f/dx^2 = \lambda f(x)$ for a real number λ. Find the eigenfunctions of D and the corresponding eigenvalues. How many linearly independent eigenfunctions are there for each eigenvalue? *Hint:* Think about exponentials.