PHYS-2106 Winter Homework 11 Due 3 April 2017

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **2nd Order Linear ODE Practice** *based on Boas*

   Solve the following differential equations for $y(x)$. If initial or boundary conditions are listed, choose integration constants to satisfy those conditions. If no intial or boundary conditions are listed, give the most general solution. Primes indicate derivatives with respect to x.

   (a) $y'' + y' - 2y = 0$ with $y(0) = 0$ and $y(1) = 1$

   (b) $y'' - 4y' + 4y = 0$

   (c) $y'' + 2y' + 10y = 100e^{i4x}$

   (d) $y'' + 9y = 0$ with $y(0) = 1$ and $y'(0) = 0$.

2. **The LRC Circuit Once Again**

   The charge on the capacitor in an electric circuit with inductance L, resistance R, and capacitance C satisfies the differential equation

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = V(t) \ , \tag{1}$$

   where $V(t)$ is the applied voltage.

   If $V(t) = 0$, show that the charge Q and current $I = dQ/dt$ are

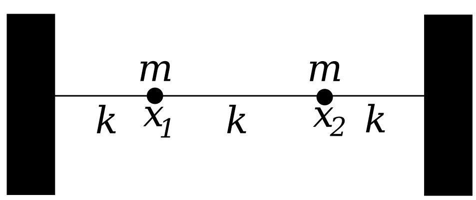
$$Q(t) = \frac{2LI_0}{\sqrt{4L/C - R^2}} e^{-Rt/2L} \sin(\omega_0 t) \ , \tag{2}$$

$$I(t) = I_0 e^{-Rt/2L} \cos(\omega_0 t) - \frac{I_0}{\sqrt{4L/R^2C - 1}} e^{-Rt/2L} \sin(\omega_0 t) \ , \tag{3}$$

   where $\omega_0 = (1/2L)\sqrt{4L/C - R^2}$, $Q(0) = 0$, and $I(0) = I_0$. Assume that $R < 2\sqrt{L/C}$.

3. **Normal Modes and Coupled Equations**

   Consider a system of 3 identical springs of spring constant k and two identical masses (mass m). As in the figure, two of the springs are attached to two walls and the two masses on the opposite ends. The two masses are joined by the third spring.



   The positions x_1 and x_2 are the displacements from the equilibrium position for each of the masses. Newton's law for the two masses can be written as

$$\frac{d^2x_1}{dt^2} = -\frac{k}{m}\left[x_1 - (x_2 - x_1)\right] = -\frac{k}{m}(2x_1 - x_2) \tag{4}$$

$$\frac{d^2x_2}{dt^2} = -\frac{k}{m}\left[x_2 + (x_2 - x_1)\right] = -\frac{k}{m}(2x_2 - x_1) \ . \tag{5}$$

(a) Define a vector $\vec{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ and write

$$\frac{d^2\vec{x}}{dt^2} = -\Omega\vec{x} \ , \tag{6}$$

where Ω is a 2×2 matrix. What is Ω?

(b) Show that a change of basis with transformation matrix S to $\vec{y} = \begin{bmatrix} y_1 & y_2 \end{bmatrix}^T = S^{-1}\vec{x}$ yields a differential equation

$$\frac{d^2\vec{y}}{dt^2} = -\Omega'\vec{y} \text{ where } \Omega' = S^{-1}\Omega S \ . \tag{7}$$

Hint: Multiply both sides by S^{-1} and insert a factor of $1 = SS^{-1}$ between Ω and $\vec{x}$.

(c) If we choose the basis to make Ω' diagonal, the differential equations become

$$\frac{d^2y_1}{dt^2} = -\Omega_1 y_1 \ , \quad \frac{d^2y_2}{dt^2} = -\Omega_2 y_2 \ , \tag{8}$$

where $\Omega_1 = 3k/m$ and $\Omega_2 = k/m$ are the eigenvalues of Ω. Find the general solution for $y_1(t)$ and $y_2(t)$. Note that each function will have two constants of integration.

(d) The eigenvectors of Ω corresponding to the eigenvalues Ω_1 and Ω_2 are respectively $\begin{bmatrix} 1 & -1 \end{bmatrix}^T/\sqrt{2}$ and $\begin{bmatrix} 1 & 1 \end{bmatrix}^T/\sqrt{2}$. Find the transformation matrix S and then find the solution for $x_1(t)$ and $x_2(t)$ using $\vec{x} = S\vec{y}$. Use initial conditions that $x_1(0) = dx_1/dt(0) = dx_2/dt(0) = 0$ and $x_2(0) = x_0$.

This is one possible useful technique for solving coupled linear equations.