PHYS-2106 Winter Homework 1 Due 16 Jan 2017

This homework is due in the dropbox outside 2L26 by 10:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. In this question, consider the usual 3D position vector space (so component indices run from 1 to 3). *Hint:* In all of the following, be careful how many different sums you have to carry out for each calculation.
   (a) Argue that $\delta_{ij}\delta_{jk} = \delta_{ik}$ and then evaluate the sum $\delta_{ij}\delta_{ij}$.
   (b) Show that $\epsilon_{ikl}\epsilon_{jkl} = 2\delta_{ij}$ and $\epsilon_{ijk}\epsilon_{abk} = \delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja}$. *Hint:* The following property may be helpful: as defined in class (and §26.8 of Riley-Hobson-Bence), ϵ_{ijk} satisfies the property that it changes sign if any pair of indices is swapped ($\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{kji} = -\epsilon_{ikj}$).
   (c) Using index notation and the ϵ symbol, prove the vector triple-product identity $\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a})$.
   (d) Using index notation and your results above, prove the "BAC-CAB" rule that $\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$.

2. Answer the following and justify your answers briefly.
   (a) Is the set of all polynomials up to order N a vector space?
   (b) Is the set of all complex numbers a real vector space? What is its dimensionality as a real vector space?
   (c) Do real functions $f(x)$ defined for $0 \le x \le 1$ with boundary conditions $f(0) = f(1) = 1$ form a vector space?
   (d) For finite complex functions $f(x)$ defined for $0 \le x \le 1$ with boundary conditions $f(0) = f(1) = 0$, is
   $$\langle f|g\rangle = \int_0^1 dx\, f(x)g(x) \tag{1}$$
   an inner product?
   (e) For finite complex functions $f(x)$ defined for $0 \le x \le 1$ with boundary conditions $f(0) = f(1) = 0$, is
   $$\langle f|g\rangle = \int_0^1 dx\, f(x)^* g(x) \tag{2}$$
   an inner product?

3. The vector $|x\rangle$ can be written as $|x\rangle = \lambda|a\rangle + \rho|b\rangle$ in a complex vector space. Show that $\langle x|y\rangle = \lambda^*\langle a|y\rangle + \rho^*\langle b|y\rangle$, where $|y\rangle$ is any other vector.

4. A 3D vector space has a basis set $\{|a_1\rangle, |a_2\rangle, |a_3\rangle\}$. These three vectors all have norm equal to 1 and inner products $\langle a_1|a_2\rangle = i/2$, $\langle a_1|a_3\rangle = 0$, and $\langle a_2|a_3\rangle = 1/\sqrt{3}$. Use the Gram-Schmidt process to find an orthonormal basis for this vector space. *Hint:* you may find the result of problem 3 to be helpful.