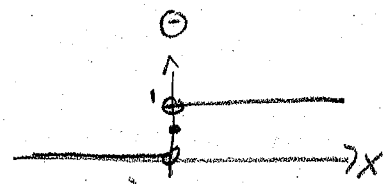
Heaviside Step and Dirac Delta functions

- Heaviside Step function
  - Define the Heaviside Step function



$$\Theta(x) = \begin{cases} 0 & x<0 \\ 1/2 & x=0 \\ 1 & x>0 \end{cases}$$

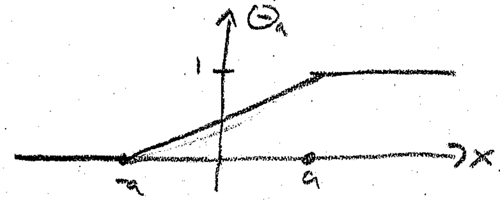
(book uses $H(x)$)

typical, but not universal.

  - This is a discontinuous function but useful for describing piece-wise defined functions

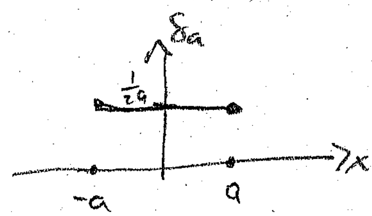
- Dirac Delta function: the derivative of Heaviside
  - Let's approach this by defining a modified step function

    + $$\Theta_a(x) = \begin{cases} 0 & x<-a \\ \left(\frac{x+a}{2a}\right) & -a<x<a \\ 1 & x>a \end{cases}$$



    + The derivative is

$$\delta_a(x) = \frac{d}{dx}\Theta_a(x) = \begin{cases} 1/2a & -a<x<a \\ 0 & |x|>a \end{cases}$$



    + $\Theta(x) = \lim_{a\to 0} \Theta_a(x)$, so we define $\delta(x) \equiv \lim_{a\to 0} \delta_a(x)$. This is zero everywhere except at $x=0$, where it is infinite. In particular, $\int_{-\infty}^{\infty} dx\, \delta_a(x) = 1$ independent of a, so define $\int_{-\infty}^{\infty} dx\, \delta(x) = 1$.
    + In fact, the integral of $\delta(x)$ from <u>any</u> negative number to <u>any</u> positive number is one.
  - The usual definition of the <u>Dirac delta function</u> (popularized by Dirac) is that $\delta(x)=0$ for $x\neq 0$ and

$$\int_{-a}^{b} dx\, \delta(x) f(x) = f(0) \text{ for } a,b>0.$$

    + You can arrive at this using a limit of any type of function that always integrates to 1 but goes to zero everywhere except $x=0$ in the limit (as above, a normalized Gaussian, etc)

+ The δ-function is not a real function. Technically speaking, it is a distribution = an instruction for doing an integral.

+ This definition is consistent with the idea that $\delta(x) = \frac{d}{dx}\Theta(x)$. Note that

$$\int_{-a}^{b} dx\, f(x) \frac{d}{dx}\Theta(x) = f(x)\Theta(x)\Big|_{-a}^{b} - \int_{-a}^{b} dx\, \Theta(x) \frac{df}{dx}$$

$$= f(b) - \int_{0}^{b} dx \frac{df}{dx} = f(0)$$

- Properties of the δ-function
  + $\int_{-\infty}^{\infty} dx\, \delta(x - x_0) = 1$, $\int_{-\infty}^{\infty} dx\, f(x)\, \delta(x - x_0) = f(x_0)$, $\delta(x) = \delta(-x)$
  + $\delta(bx) = \frac{1}{|b|}\delta(x)$ b/c $\int_{-\infty}^{\infty} dx\, f(x)\, \delta(bx) = \int_{-\infty}^{\infty} \frac{dx'}{|b|} f(\frac{x'}{b})\, \delta(x') = \frac{1}{|b|} f(0)$
  + Similarly, $\delta(h(x)) = \sum_i \delta(x - x_i)/|h'(x_i)|$ where x_i are the zeros of $h(x)$.

- Physical applications of the δ-function
  + We can use a δ-function to represent any infinitely localized source, such as a sharp kick as $\delta(t)$
  + A 3D δ-function $\delta^3(\vec{x}) = \delta(x)\delta(y)\delta(z)$ represents a point charge at the origin: $\rho(\vec{x}) = q\,\delta^3(\vec{x})$ is the charge density, etc
  + The units of a δ-function are 1/the units of the argument.

  + Integration by parts allows us to understand derivatives of δ:

$$\int_{-\infty}^{\infty} dx\, f(x) \frac{d}{dx}\delta(x) = -\int_{-\infty}^{\infty} dx\, \delta(x) \frac{df}{dx}(x) = -\frac{df}{dx}(0)$$