Some Special Integral-Related Functions

Gaussian Integrals (Gaussian function $\propto e^{-\alpha x^2}$)

- Can you carry out an integral of the form $\int dx\, e^{-\alpha x^2} x^n$?
  There is no closed indefinite integral for n even.
  - Start with $n=1$. Substitute $u=x^2$, so $\int dx\, x e^{-\alpha x^2} = \frac{1}{2}\int du\, e^{-\alpha u}$
    $= \frac{-1}{2\alpha} e^{-\alpha x^2}$.
  - We can't do an indefinite integral of the Gaussian, but consider
    $I(\alpha) = \int_{-\infty}^{\infty} dx\, e^{-\alpha x^2}$
    + Then $I(\alpha)^2 = \left(\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2}\right)\left(\int_{-\infty}^{\infty} dy\, e^{-\alpha y^2}\right) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, e^{-\alpha(x^2+y^2)} = \int_0^{2\pi} d\theta \int_0^{\infty} dr\, r e^{-\alpha r^2}$
    + We see $I(\alpha)^2 = (2\pi)\left[\frac{-1}{2\alpha} e^{-\alpha r^2}\right]_0^{\infty} = \pi/\alpha \Rightarrow I(\alpha) = \sqrt{\pi/\alpha}$
    + Of course $\int_0^{\infty} dx\, e^{-\alpha x^2} = \frac{1}{2}\sqrt{\pi/\alpha}$
  - For higher n, we can evaluate the integral by differentiation
    + $\int_0^{\infty} dx\, x e^{-\alpha x^2} = \frac{1}{2\alpha} \Rightarrow \int_0^{\infty} dx\, x^{2n+1} e^{-\alpha x^2} = (-1)^n \frac{d^n}{d\alpha^n}\left(\int_0^{\infty} dx\, x e^{-\alpha x^2}\right) = \frac{n!}{2\alpha^{n+1}}$
    + And $\int_0^{\infty} dx\, x^{2n} e^{-\alpha x^2} = (-1)^n \frac{d^n}{d\alpha^n}\left(\int_0^{\infty} dy\, e^{-\alpha x^2}\right) = \frac{1}{2}\left(\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2n-1}{2}\right)\sqrt{\frac{\pi}{\alpha^{2n+1}}}$
  - Gaussians are normalized in various ways in different applications
    + Quantum mechanics Gaussian wavefunction $\psi(x) = (\alpha/\pi)^{1/4} e^{-\alpha x^2/2}$
      is normalized so $\int_{-\infty}^{\infty} dx\, |\psi(x)|^2 = 1$
    + Statistics: the "standard" Gaussian distribution is
      $f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ normalized so $\int_{-\infty}^{\infty} dz\, f(z) = 1$.
- Error function: since the Gaussian has no closed form indefinite integral, there is no closed form definite integral with finite limits
  - We define the error function $\mathrm{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x du\, e^{-u^2}$
  - This is normalized so $\mathrm{erf}(\infty) = 1$. Note that $\mathrm{erf}(0) = 0$
    and $\mathrm{erf}(-x) = -\mathrm{erf}(x)$.

- Gamma Function

- A generalized factorial function
  - Define the gamma function $\Gamma(z) = \int_0^\infty dx\, x^{z-1} e^{-x}$ for $z > 0$.
  - Integration by parts gives a recursion relation:
    + $\Gamma(z+1) = \int_0^\infty dx\, x^z e^{-x} = -x^z e^{-x}\Big|_0^\infty + z\int_0^\infty dx\, x^{z-1} e^{-x} = z\Gamma(z)$
    + Furthermore, $\Gamma(1) = 1$ by direct integration $\Rightarrow$
    + For nonnegative integer n, $\Gamma(n+1) = n!$
  - In fact, complex analysis says that there is a unique way to define $\Gamma(z)$ for complex z except at $z = 0, -1, -2, \ldots$ where it diverges.
  - To evaluate at half-integer arguments
    + First, change integration variables using $x = y^2$, so $\Gamma(z) = 2\int_0^\infty dy\, y^{2z-1} e^{-y^2}$
    + Then evaluate $\Gamma(1/2)$ as a Gaussian integral $\Gamma(1/2) = \sqrt{\pi}$
    + Use the recursion relation $\Gamma(-\frac{1}{2}) = -2\Gamma(\frac{1}{2}) = -2\sqrt{\pi}$, etc and $\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi}$, etc.
  - Stirling's Approximation. Factorials grow very quickly. How quickly?
    + Write $n! = \Gamma(n+1) = \int_0^\infty dx\, x^n e^{-x} = \int_0^\infty dx\, e^{n\ln x - x}$
    + The integrand is maximized when $(nx^{n-1}e^{-x} - x^n e^{-x}) = 0$, or $x = n$.
      So write $x = n+y$, so $\ln x = \ln n + \ln(1+\frac{y}{n}) = \ln n + \frac{y}{n} - \frac{y^2}{2n^2} + \ldots$
    + Then
      $$n! = \int_{-n}^\infty dy\, \exp\left[n\left(\ln n + \frac{y}{n} - \frac{y^2}{2n^2} + \ldots\right) - n - y\right]$$
      Most of the value comes from $|y/n| \ll 1$, so drop $\ldots$ terms.
    + For n large, we can take the lower limit to $-\infty$.
      Then
      $$n! \approx e^{n\ln n - n}\int_{-\infty}^\infty dy\, e^{-y^2/2n} = \sqrt{2\pi n}\, n^n e^{-n} \text{ - Stirling's approximation}$$

- Related Functions
  - The beta function is $B(m,n) = \int_0^1 dx\, x^{m-1}(1-x)^{n-1}$
    - You can re-write in various ways by changing integration variables including $x = \sin^2\theta \implies B(m,n) = 2\int_0^{\pi/2} d\theta\, (\sin\theta)^{2m-1}(\cos\theta)^{2n-1}$
    - Then we can re-write in Gaussian form

$$\Gamma(n)\Gamma(m) = 4\int_0^\infty dx \int_0^\infty dy\; x^{2n-1}\, y^{2m-1}\, e^{-(x^2+y^2)}$$

$$= 4\int_0^{\frac{\pi}{2}} d\theta \int_0^\infty dr\; r^{2m+2n-1}(\cos\theta)^{2n-1}(\sin\theta)^{2m-1}e^{-r^2}$$

$$= B(m,n)\,\Gamma(m+n) = (\text{angular})\times(\text{radial})$$

    - The beta function shows up in integrals in advanced physics
  - The incomplete gamma functions are finite definite integrals
    - $\gamma(z,x) \equiv \int_0^x du\; u^{z-1}e^{-u}$ and $\Gamma(z,x) \equiv \int_x^\infty du\; u^{z-1}e^{-u}$
    - Then $\Gamma(z) = \gamma(z,x) + \Gamma(z,x)$