Fourier transforms

— Fourier series give an alternate definition for functions that are periodic or given on a finite interval. What about nonperiodic functions defined for the whole real line?

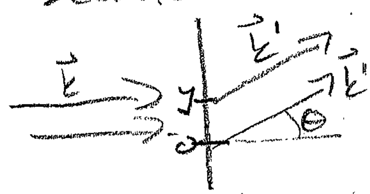
- Let's start with a function of period L and then take $L \to \infty$
  - + We can write the complex Fourier series
$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp[ik_n x] \quad \text{with wavenumber } k_n \equiv \frac{2\pi n}{L}$$
  - + The coefficient is
$$c_n = \frac{1}{L}\int_{-L/2}^{L/2} dx\, f(x)\, e^{-ik_n x} = \frac{\Delta k}{2\pi}\int_{-L/2}^{L/2} dx\, f(x)\, e^{-ik_n x}$$
    in terms of the change Δk in wavenumbers from n to $n+1$.
  - + Then, as $L \to \infty$, $\Delta k \to 0$, so
$$f(x) = \sum_n \frac{\Delta k}{2\pi} e^{ik_n x}\int_{-L/2}^{L/2} dx'\, f(x')\, e^{-ik_n x'} \longrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx}\left(\int_{-\infty}^{\infty} dx'\, f(x')\, e^{-ikx'}\right)$$
- The Fourier series has become an integral:
  - + We define the <u>Fourier transform</u> $\tilde{f}(k)$ of $f(x)$
$$\tilde{f}(k) \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, f(x)\, e^{-ikx} \quad \left(\text{also sometimes called } \mathcal{F}[f](k)\right)$$
  - + The original function is given by the <u>inverse Fourier transform</u>
$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, \tilde{f}(k)\, e^{+ikx}$$
  - + Some authors have other conventions for the $1/2\pi$ factor or the sign in the complex exponential. This is the RHB convention and is the most common one in physics.
- Work examples (Gaussian, hat function)

- A particularly important example is the δ-function:
  - $\mathcal{F}[\delta](k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, \delta(x) e^{ikx} = \frac{1}{\sqrt{2\pi}}$ = constant!
  - The inverse transform gives a new definition for the δ-function

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx}$$

Some uses in physics

- Quantum Mechanics
  - From de Broglie, momentum $p = h/\lambda = \hbar k$ in terms of wave number
  - The F.T. lets us write the wavefunction in terms of momentum in other words, $\tilde{\psi}(k)$ has all the information of the wavefunction but is a function of momentum
  - From our examples, we see that the more localized $\psi(x)$ is, the more spread out $\tilde{\psi}(p)$ will be + vice versa.
  - In fact $\Delta k \Delta x \gtrsim 1 \Rightarrow \Delta p \Delta x \gtrsim \hbar \Rightarrow$ this is the basis for the Heisenberg uncertainty principle..
- Diffraction in optics
  - Imagine light hits a screen that's only partly transparent and is scattered.



Light from different parts of the screen with the same $\vec{k}'$ go off to a far away detector at $\vec{r}$ at the same angle θ.

  - Let's say the screen 'takes' an incoming wave of amplitude A and turns it into $Af(y)$ as it leaves. Then the plane wave at position $\vec{r}$ emitted from point y on the screen is $\sim Af(y)\exp[i\vec{k}'\cdot(\vec{r} - y\hat{j})]$
  - If we add up over the whole screen, the wave we see at the detector is $\propto \int_{-\infty}^{\infty} dy\, f(y)\, e^{-iky\sin\theta} = \tilde{f}(k\sin\theta)$.

Spectra

+ Suppose a radio (telescope) receives a wave of amplitude $f(t)$. The F.T. $\tilde{f}(\omega)$ tells us the amplitude in each frequency mode
+ For a TV or radio signal, $\tilde{f}(\omega)$ is peaked around the station's assigned frequency.
+ Energy in an EM wave is amplitude squared, so $|\tilde{f}(\omega)|^2$ tells us the <u>energy spectrum</u> of the wave.
+ Can apply to energy as a function of time → spectrum or the wavelength spectrum of an image, etc.

Some properties & theorems

- For an odd or even function $f(x)$, the F.T. simplifies because $e^{ikx} = \cos(kx) + i\sin(kx)$.
  + For an even function
  $$\mathcal{F}[f](k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, f(x)\left[\cos(kx) - i\sin(kx)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, f(x)\cos(kx)$$
  $$\equiv \mathcal{F}_c[f](k) = \text{Fourier cosine transform}$$
  + Odd functions have the <u>Fourier sine transform</u>
  $$\mathcal{F}_s[f](k) \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, f(x)\sin(kx) \equiv +i\mathcal{F}[f](k)$$
  + The inverses of the sine + cosine transforms are the same as the transforms themselves.

- Some basic properties (that we can see as examples).
  + Derivatives: $\mathcal{F}\left[\frac{df}{dx}\right](k) = +ik\tilde{f}(k)$; $\mathcal{F}^{-1}\left[\frac{d\tilde{f}}{dk}\right](x) = -ixf(x)$
  + and so on for higher derivatives
  + If $f(x)$ is real, $\tilde{f}(k)^* = \tilde{f}(-k)$
  + Integration: $\mathcal{F}\left[\int_a^x dx' f(x')\right] = \frac{1}{ik}\tilde{f}(k) + c\,\delta(k)$

  If $a \to -\infty$, $c = \sqrt{2\pi}\int_{-\infty}^{\infty} dx\, f(x)$.

+ Scaling + shifts: $\mathcal{F}[f(ax)] = \frac{1}{a}\tilde{f}(\frac{k}{a})$, $\mathcal{F}[f(x+a)] = e^{ika}\tilde{f}(k)$,

and $\mathcal{F}[e^{-i\alpha x}f(x)] = \tilde{f}(k+\alpha)$ (α may be complex)

Convolutions

+ The <u>convolution</u> of 2 functions $f(x)$ and $g(x)$ is defined

$$f*g(x) = \int_{-\infty}^{\infty} dy\, f(y)\, g(x-y)$$ (← careful of notation!)

The convolution is commutative (prove!), associative (prove!), + distributive.

+ An example is the image from a telescope. We say the light hitting the camera is from where the scope is pointing, but really other spots on the lens contribute. So the amount of light at pt. x is a convolution of the true image w/ the transfer function.

+ <u>Convolution theorem</u>: $\mathcal{F}[f*g] = \sqrt{2\pi}\,\tilde{f}(k)\tilde{g}(k)$; $\mathcal{F}^{-1}[\tilde{f}*\tilde{g}] = \sqrt{2\pi}\, f(x)g(x)$

Proof of 1st equality:

$$\mathcal{F}[f*g](k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-ikx}\int_{-\infty}^{\infty} dy\, f(y)g(x-y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dy\, f(y)\int_{-\infty}^{\infty} dx\, g(x-y)e^{-ikx}$$

If $u = x-y$, we have

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dy\, f(y)e^{-iky}\int_{-\infty}^{\infty} du\, g(u)e^{-iku} = \sqrt{2\pi}\,\tilde{f}(k)\tilde{g}(k)\ \checkmark$$

+ If you know a measurement device's transfer function, you can use F.T.s to get the true image!

Correlation functions

+ The <u>correlation</u> of 2 functions is a measure of how similar they are a distance apart, defined

$$[f\otimes g](x) = \int_{-\infty}^{\infty} dy\, f(y)^* g(y+x)$$

+ This is associative + distributive, but $[f\otimes g](x) = [g\otimes f]^*(-x)$

+ Then the convolution theorem tells us

$$\mathcal{F}[f\otimes g] = \sqrt{2\pi}\,\tilde{f}(k)^*\tilde{g}(k)$$

+ The <u>power spectrum</u> or <u>energy spectrum</u> of an amplitude function $f(x)$ is the FT of the correlation of $f(x)$ with itself.

- There is also a version of Parseval's Theorem for Fourier transforms:
$$\int_{-\infty}^{\infty} dx\, |f(x)|^2 = \int_{-\infty}^{\infty} dk\, |\tilde{f}(k)|^2 \qquad \text{(Prove)}$$

- Multiple dimensions:
  + The Fourier transform for a function of several variables is the F.T. for each variable:
$$\tilde{f}(k_1, k_2, k_3 \cdots) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx_1\, e^{-ikx_1} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx_2\, e^{-ik_1 x_2} \left(\cdots f(x_1, x_2, \cdots) \right) \right)$$
etc.
  + In 3D, we can add the exponents together to find
$$\tilde{f}(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int d^3\vec{x}\, f(\vec{x}) e^{-i\vec{k}\cdot\vec{x}}, \qquad f(\vec{x}) = \frac{1}{(2\pi)^{3/2}} \int d^3\vec{k}\, \tilde{f}(k) e^{+i\vec{k}\cdot\vec{x}}$$
  + The 3D δ-function is $\delta^3(\vec{x}) = \frac{1}{(2\pi)^3} \int d\vec{k}\, e^{i\vec{k}\cdot\vec{x}}$