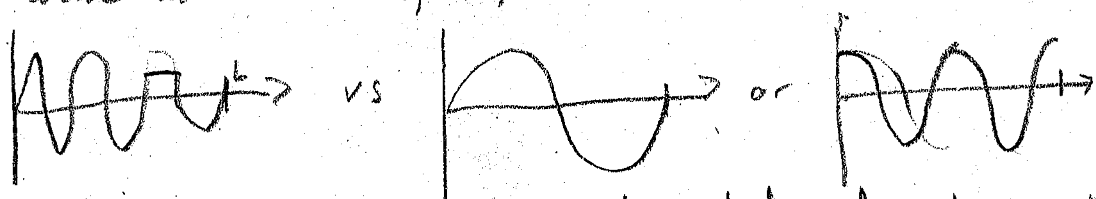
Fourier Analysis

Fourier Series

- Periodic motion / behavior is very common in physics:
  - Examples include orbits, electrical signals, sounds of a particular pitch, density in a crystal, etc
  - Mathematically, $f(x)$ has period L if $f(x) = f(x+L)$ for every x.
  - Although the periodic function can have any shape, for sinusoidal behavior (like simple harmonic motion) have special importance.
    + These are "modes" of a specific (angular) frequency (in time) or wave number (in space)



    + For example, these might be the fundamental mode + harmonics of a string
  - Our goal is to write a general periodic function in terms of sinusoidal functions of definite frequencies / wavenumbers
- The Fourier series for a function of period L
  - A periodic function can be written as

$$f(x) = \left\{ \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left[\frac{2\pi n x}{L}\right] \right\} + \sum_{n=1}^{\infty} b_n \sin\left[\frac{2\pi n x}{L}\right]$$

  The sum converges to $f(x)$ everywhere that $f(x)$ is continuous if (Dirichlet conditions)
    + $f(x)$ is periodic
    + $f(x)$ is single-valued (ie, for each x, there is a single value $f(x)$) and continuous except at a finite # of finite discontinuities
    + $f(x)$ has a finite number of local maxima + minima in a period
    + the integral of $|f(x)|$ over a period converges

- How do we find the coefficients?
  - \+ We note that (for $n, m > 0$ integers)

$$\int_0^L dx\, \sin\left(\frac{2\pi n x}{L}\right) \cos\left(\frac{2\pi m x}{L}\right) = 0$$

calculate

$$\int_0^L dx\, \cos\left(\frac{2\pi n x}{L}\right)\cos\left(\frac{2\pi m x}{L}\right) = \frac{1}{2} L\, \delta_{n,m}\ ;\quad \int_0^L dx\, \sin\left(\frac{2\pi n x}{L}\right)\sin\left(\frac{2\pi m x}{L}\right) = \frac{1}{2} L\, \delta_{m,n}$$

as well as $\int_0^L dx\, \cos\left(\frac{2\pi n x}{L}\right) = \int_0^L dx\, \sin\left(\frac{2\pi n x}{L}\right) = 0$ and $\int_0^L dx = L$.

  - \+ So plug in the Fourier series

$$\int_0^L dx\, f(x) = \frac{a_0}{2} L + \sum_n a_n \int_0^L dx\, \cos\left(\frac{2\pi n x}{L}\right) + \sum_n b_n \int_0^L dx\, \sin\left(\frac{2\pi n x}{L}\right) = \frac{a_0 L}{2}$$

$$\int_0^L dx\, \cos\left(\frac{2\pi n x}{L}\right) f(x) = \sum_m a_m \int_0^L dx\, \cos\left(\frac{2\pi n x}{L}\right)\cos\left(\frac{2\pi m x}{L}\right) = \frac{a_n L}{2}$$

$$\int_0^L dx\, \sin\left(\frac{2\pi n x}{L}\right) f(x) = \sum_m b_m \int_0^L dx\, \sin\left(\frac{2\pi n x}{L}\right)\sin\left(\frac{2\pi m x}{L}\right) = \frac{b_n L}{2}.$$

  - \+ In other words (for $n \geq 0$)

$$a_n = \frac{2}{L}\int_0^L dx\, f(x) \cos\left(\frac{2\pi n x}{L}\right), \quad b_n = \frac{2}{L}\int_0^L dx\, f(x) \sin\left(\frac{2\pi n x}{L}\right)$$

  - \+ Because $f(x)$ is periodic, you can shift the limits of integration as long as they cover one period. Integrating from $-\frac{L}{2}$ to $\frac{L}{2}$ is often useful.

- Note that $f(x)$ can take complex values if a_n and b_n are complex. You can see this by saying $f(x) = f_1(x) + i f_2(x)$ with $f_1(x)$ and $f_2(x)$ real and taking Fourier series of f_1 & f_2.

- Even & odd functions have simplified Fourier series
  - \+ For example, if $f(x)$ is even, then $b_n = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx\, f(x) \sin\left(\frac{2\pi n x}{L}\right) = 0$
  - \+ For odd functions, $a_n = 0$; for even functions, $b_n = 0$.

- Integration: Since integration is linear,

$$\int_0^x dx' f(x') = \frac{a_0}{2}x + \sum_n \frac{a_n L}{2\pi n}\sin\left[\frac{2\pi n x}{L}\right] - \sum_n \frac{b_n L}{2\pi n}\cos\left[\frac{2\pi n x}{L}\right]$$

  + If $a_0 = 0$, the RHS is already a Fourier series
  + If $a_0 \neq 0$, we have the Fourier series for $\int_0^x dx' f(x') - \frac{a_0 x}{2}$.

    Or we can replace x by its Fourier series (see below)

- Differentiation: if $f(x)$ is continuous and $f'(x)$ satisfies the Dirichlet conditions, the Fourier series of $f'(x)$ is the derivative of the series for $f(x)$:

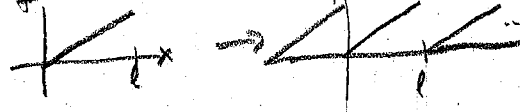
$$f'(x) = -\sum_n \frac{2\pi n a_n}{L}\sin\left[\frac{2\pi n x}{L}\right] + \sum_n \frac{2\pi n b_n}{L}\cos\left[\frac{2\pi n x}{L}\right]$$

  Note: there can be convergence problems for this series if $f(x)$ is not everywhere continuous

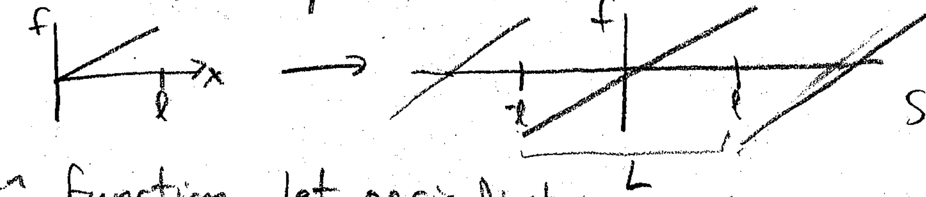
– Applications:

- Discontinuous functions
  + A Taylor series is not valid past a discontinuity, so Fourier series have an advantage here
  + At a discontinuity in $f(x)$, the Fourier series takes a value half-way between upper + lower values
  + Gibbs phenomenon: at a discontinuity, a finite # of terms always "overshoots" the discontinuity. As the terms add up, the overshoot remains but moves closer to the discontinuity (so it eventually only happens in a vanishing space)
  + Square wave example
- Non-periodic functions
  + What if we only have $f(x)$ defined on a finite range, say $0 \leq x \leq l$? There are a few ways to arrive at a Fourier series for $f(x)$ that give the same values for $0 \leq x \leq l$

+ First, choose periodicity $L = l$. Then define $f(x+kL) = f(x)$ for $0 \le x \le L$ and k any integer

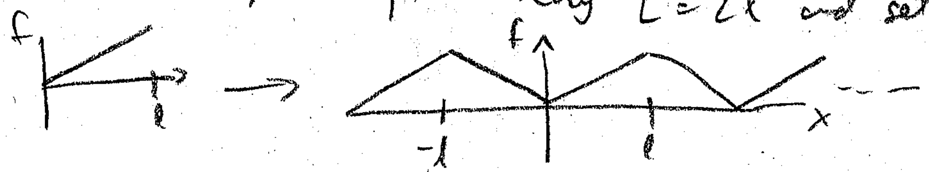


+ We could instead take advantage of simplifications for odd/even functions. For an odd function, let periodicity $L = 2l$ and choose $f(-x) = -f(x)$.



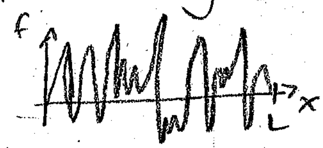
Series has only sine terms

+ For an even function, let periodicity $L = 2l$ and set $f(-x) = f(x)$

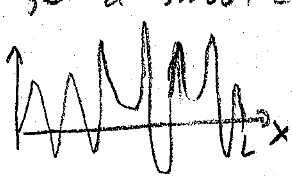


Series has constant + cosine terms

- Filtering / Compression
  + Suppose you have a function (like the intensity of a black/white image) with a lot of fine detail



  + You can get roughly the same picture by (1) taking the Fourier series (same information) and (2) dropping the high wave number terms (ie, $n >$ some number). You get a smoothed out picture but requiring less resolution / storage



  + The same idea works with sound or electrical signals. The function may be a function of time. For discontinuous functions, this type of filter smooths out the discontinuities
  + The same idea applies to Fourier transforms we will talk about later.

— Complex Fourier Series

- Remember that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

So we can re-write the Fourier series as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2\pi i n x}{L}\right) \quad \text{with} \quad \left.\begin{array}{l} c_n = \frac{1}{2}(a_n - i b_n) \\ c_{-n} = \frac{1}{2}(a_n + i b_n) \end{array}\right\} \begin{array}{l} n \geq 0 \\ (b_0 \equiv 0) \end{array}$$

- We can start directly with the complex exponential version
  + We see $$\int_0^L dx\, e^{-i\frac{2\pi m x}{L}} e^{i\frac{2\pi n x}{L}} = L\,\delta_{m,n} \quad (m,n = \text{any integers})$$
  + That means $$c_n = \frac{1}{L}\int_0^L dx\, e^{-i\frac{2\pi n x}{L}} f(x)$$

    Of course, you can shift the integration limits as before.
- For a real function $f(x)$, we have $f(x)^* = f(x) \Rightarrow$
  + $$\sum_n c_n^* \exp\left(-\frac{2\pi i n x}{L}\right) = \sum_n c_n \exp\left(\frac{2\pi i n x}{L}\right)$$
  + Multiply by $e^{+2\pi i m x/L}$ on both sides + integrate $\int_0^L dx$. Then we find that $c_m^* = c_{-m}$.
- Suppose we have 2 periodic functions $f(x)$ and $g(x)$ with complex Fourier coefficients c_n and γ_n respectively
  + Then $$\frac{1}{L}\int_0^L dx\, f(x)^* g(x) = \frac{1}{L}\sum_n\sum_m \int_0^L dx\, c_n^* e^{-\frac{i2\pi n x}{L}} \gamma_m e^{+\frac{i2\pi m x}{L}} = \sum_{n=-\infty}^{\infty} c_n^* \gamma_n$$
  + This looks a lot like an inner product: with $\frac{1}{L}\int_0^L dx\, f^*(x) g(x)$ as an inner product, the complex exponentials (and sine/cosine) make an orthonormal basis (in ∞ dimensions)
  + Parseval's theorem: take $g(x) = f(x)$, so $$\frac{1}{L}\int_0^L dx\, |f(x)|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2 \quad \left(= \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) \text{ for a real function } f(x).\right)$$
  + Sometimes Parseval's theorem (or a general Fourier series) can be used to carry out a particular sum.