Mathematical Physics II PHYS-2106
Final Examination

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Instructions:

- Do not turn over until instructed.
- You will have 3 hours to complete this test.
- No electronic devices or hardcopy notes are allowed.
- INSTRUCTIONS REGARDING TEST LENGTH WILL GO HERE.
- **Answer all questions briefly and completely; you can earn partial credit if you show your work.**
- **Only the lined pages of your exam book will be graded. Use the blank pages for scratch work only.**

Useful Formulae

Linear Algebra

- Trace: defined as $\text{tr}(A) = A_{ii}$; linear $\text{tr}(A+B) = \text{tr}(A) + \text{tr}(B)$; cyclic $\text{tr}(AB) = \text{tr}(BA)$
- Determinant:
  - $\det(\lambda A) = \lambda^N \det(A)$, $\det(AB) = \det(A)\det(B)$
  - Changes sign if you reverse two rows or columns
  - Unchanged if you add a multiple of one row (column) to another row (column)
- General formula for inverse: $A^{-1} = \text{cof}(A)^T / \det(A)$; $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
- Change of basis: columns of S are components of new basis with respect to old basis
  - Vector components $x' = S^{-1}x$
  - Similarity transformation for matrix components $A' = S^{-1}AS$

Special Functions

- Gaussian integrals
$$\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}\ , \quad \int_{-\infty}^{\infty} dx\, x e^{-\alpha x^2} = \frac{1}{2\alpha}$$
- Gamma and related functions
  - $\Gamma(z) = \int_0^\infty du\, u^{z-1} e^{-u}$ $\Gamma(z+1) = z\Gamma(z)$, $\Gamma(n+1) = n!$ for $n \geq 0$ integer, $\Gamma(1/2) = \sqrt{\pi}$
  - Stirling's approximation $\Gamma(n+1) \approx \sqrt{2\pi n} n^n e^{-n}$
  - Beta function
$$B(m,n) = \int_0^1 dx\, x^{m-1}(1-x)^{n-1} = 2\int_0^{\pi/2} d\theta\, (\sin\theta)^{2m-1}(\cos\theta)^{2n-1} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$
- Heaviside function $\Theta(x) = 1$ for $x > 0$, $= 0$ for $x < 0$, and $\Theta(0) = 1/2$

- Dirac δ function $\int_{-a}^{b} dx\, \delta(x) f(x) = f(0)$ for $a, b > 0$; other properties by change of variables, integration by parts

Fourier Analysis

- Fourier Series for function of period L
  - Real Series
$$f(x) = \left\{ \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{L} x\right) \right\} + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n}{L} x\right)$$
$$a_n = \frac{2}{L} \int_0^L dx\, f(x) \cos\left(\frac{2\pi n}{L} x\right) \;\;, \quad b_n = \frac{2}{L} \int_0^L dx\, f(x) \sin\left(\frac{2\pi n}{L} x\right)$$
  - Complex Series
$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2\pi i n}{L} x\right) \;\;, \quad c_n = \frac{1}{L} \int_0^L dx\, f(x) \exp\left(-\frac{2\pi i n}{L} x\right)$$
  - Parseval's theorem
$$\frac{1}{L} \int_0^L |f(x)|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2$$
  - Limits of integration can be shifted as long as the integral covers exactly one period
- Fourier transform
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, \tilde{f}(k) e^{ikx} \;\;, \quad \tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, f(x) e^{-ikx}$$
  - $\mathcal{F}[df/dx] = ik\tilde{f}(k)$ $\mathcal{F}^{-1}[d\tilde{f}/dk] = -ixf(x)$,
  - Convolution
$$f \star g(x) = \int_{-\infty}^{\infty} dy f(y) g(x-y) \;\;, \quad \mathcal{F}[f \star g](k) = \sqrt{2\pi} \tilde{f}(k) \tilde{g}(k)$$
  - Correlation
$$f \otimes g(x) = \int_{-\infty}^{\infty} dy f(y)^* g(x+y) \;\;, \quad \mathcal{F}[f \otimes g](k) = \sqrt{2\pi} \tilde{f}(k)^* \tilde{g}(k)$$
  - Parseval's theorem
$$\int_{-\infty}^{\infty} dx\, |f(x)|^2 = \int_{-\infty}^{\infty} dk\, |\tilde{f}(k)|^2$$

Differential Equations

- first-order: consider separable variables and exact equations
- linear first-order: $y' + P(x) y = Q(x) \Rightarrow \; y = e^{-I} \int dx e^{I} Q + A e^{-I}$ with $I = \int dx P$.
- second-order linear: exponential solution, auxiliary equation for homogeneous

Some trig and hyperbolic trig identities

- $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$, $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i$, $\cosh\theta = (e^{\theta} + e^{-\theta})/2$, $\sinh\theta = (e^{\theta} - e^{-\theta})/2$
- $\cos^2\theta + \sin^2\theta = \cosh^2\theta - \sinh^2\theta = 1$
- $\sin(2\theta) = 2\sin\theta\cos\theta$, $\cos(2\theta) = \cos^2\theta - \sin^2\theta$, $\sinh(2\theta) = 2\sinh\theta\cosh\theta$, $\cosh(2\theta) = \cosh^2\theta + \sinh^2\theta$
- other angle addition formulas, etc, follow from exponential definition