Mathematical Physics II PHYS-2106
Second In-Class Test

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Instructions:

- Do not turn over until instructed.
- You will have 50 minutes to complete this test.
- No electronic devices or hardcopy notes are allowed.
- INSTRUCTIONS REGARDING TEST LENGTH WILL GO HERE.
- **Answer all questions briefly and completely; you can earn partial credit if you show your work.**
- **Only the lined pages of your exam book will be graded. Use the blank pages for scratch work only.**

Useful Formulae

- Change of basis
  - Transformation matrix S: columns are components of new basis with respect to old basis
  - Vector components $x' = S^{-1}x$
  - Starting from an orthonormal basis, the new basis is orthonormal if the transformation matrix is unitary
  - Similarity transformation for matrix components $A' = S^{-1}AS$
  - Diagonal form has eigenvalues on diagonal, basis of eigenvectors
  - Normal matrices $AA^\dagger = A^\dagger A$; include (anti-)Hermitian $A^\dagger = \pm A$ and unitary $A^\dagger = A^{-1}$; unitarily diagonalizable
- Gaussian integrals
$$\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}\ , \quad \int_{-\infty}^{\infty} dx\, x e^{-\alpha x^2} = \frac{1}{2\alpha}$$
  - Related integrals by changing limits, differentiating w.r.t. parameter
  - Error function: $\mathrm{erf}(x) = (2/\sqrt{\pi})\int_0^x du\, \exp(-u^2),\ \mathrm{erf}(\infty) = 1$
- Gamma and related functions (for $z > 0$)
  - $\Gamma(z) = \int_0^\infty du\, u^{z-1}e^{-u}\ ;\ \gamma(z,x) = \int_0^x du\, u^{z-1}e^{-u}\ ;\ \Gamma(z,x) = \int_x^\infty du\, u^{z-1}e^{-u}$
  - $\Gamma(z+1) = z\Gamma(z)$, $\Gamma(n+1) = n!$ for $n \geq 0$ integer, $\Gamma(1/2) = \sqrt{\pi}$
  - Stirling's approximation $\Gamma(n+1) \approx \sqrt{2\pi n}n^n e^{-n}$
  - Beta function
$$B(m,n) = \int_0^1 dx\, x^{m-1}(1-x)^{n-1} = 2\int_0^{\pi/2} d\theta\, (\sin\theta)^{2m-1}(\cos\theta)^{2n-1} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

- Heaviside function $\Theta(x) = 1$ for $x > 0$, $= 0$ for $x < 0$, and $\Theta(0) = 1/2$
- Dirac δ function $\int_{-a}^{b} dx\, \delta(x) f(x) = f(0)$ for $a, b > 0$
- Fourier Series for function of period L
  - Dirichlet conditions (finite number of finite discontinuities, finite number of extrema, single-valued, integral of $|f(x)|$ converges)
  - Real Series

$$f(x) = \left\{ \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{L} x\right) \right\} + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n}{L} x\right)$$

$$a_n = \frac{2}{L} \int_0^L dx\, f(x) \cos\left(\frac{2\pi n}{L} x\right) \quad , \quad b_n = \frac{2}{L} \int_0^L dx\, f(x) \sin\left(\frac{2\pi n}{L} x\right)$$

  - Complex Series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2\pi i n}{L} x\right) \quad , \quad c_n = \frac{1}{L} \int_0^L dx\, f(x) \exp\left(-\frac{2\pi i n}{L} x\right)$$

  - Parseval's theorem

$$\frac{1}{L} \int_0^L |f(x)|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2$$

  - Limits of integration can be shifted as long as the integral covers exactly one period
- Fourier transform

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, \tilde{f}(k) e^{ikx} \quad , \quad \tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, f(x) e^{-ikx}$$

- Some trig and hyperbolic trig identities
  - $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$, $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i$, $\cosh\theta = (e^{\theta} + e^{-\theta})/2$, $\sinh\theta = (e^{\theta} - e^{-\theta})/2$
  - $\cos^2\theta + \sin^2\theta = \cosh^2\theta - \sinh^2\theta = 1$
  - $\sin(2\theta) = 2\sin\theta\cos\theta$, $\cos(2\theta) = \cos^2\theta - \sin^2\theta$, $\sinh(2\theta) = 2\sinh\theta\cosh\theta$, $\cosh(2\theta) = \cosh^2\theta + \sinh^2\theta$
  - other angle addition formulas, etc, follow from exponential definition