Other Topics

- Eigenfunction/eigenvalue problems
  - Consider the linear ODE $a(x)\frac{d^2y}{dx^2} + b(x)\frac{d^2y}{dx} + c(x)\,y = f(x)$
    - + The left-hand side is a <u>linear operator</u> on the vector space of functions.
    - + That is, if we write the equation as $Dy = f$, $D(y+z) = Dy + Dz$, and $D(\lambda y) = \lambda Dy$ for $\lambda =$ constant
    - + Our ODE can be written as a vector equation! $D\hat{y} = f$ has the same structure as the matrix eqn $Ax = b$
  - Just like with matrices, the <u>differential operator</u> D has <u>eigenvalues</u> and <u>eigenfunctions</u> (eigenvectors).
    - + An eigenfunction of D is a function $y(x)$ such that $Dy = \lambda y$ where $\lambda =$ constant is the eigenvalue
    - + Finding eigenfunctions + eigenvalues requires solving the ODE and applying any boundary conditions
  - Example: time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\,\psi = \underset{\substack{\uparrow \\ \text{energy}}}{E}\,\psi \implies \underset{\text{operator}}{\left[\frac{d^2}{dx^2} - \hat{V}(x)\right]}\psi(x) = \underset{\text{eigenvatue}}{\left(\frac{-2mE}{\hbar^2}\right)}\psi(x)$$

    - + For energy $E > 0$, and $V(x) = 0$, we have $\frac{d^2\psi}{dx^2} = -k^2\psi(x)$
    - + The general solution can be written

$$\psi(x) = A_1\cos(kx) + B_1\sin(kx) \quad \text{or} \quad \psi(x) = A_2 e^{ikx} + B_2 e^{-ikx}$$

    - + Allowed values of k + therefore the energy eigenvalue depend on the boundary conditions. Consider Dirichlet, Neumann, + periodic

Series Solutions (aka Method of Frobenius)

- We've just looked at 2nd order ODEs with constant coefficients. These are solved by real + complex exponentials. + If the coefficients of d^2y/dx^2, dy/dx, and y depend on x, the solutions are <u>special functions</u>
  - \+ Example: $x^2y'' + xy' + (x^2 - \nu^2)y = 0$ is <u>Bessel's equation</u>; $y(x)$ = Bessel function
- If we don't know the specific equation, we can solve for $y(x)$ as a power series. General idea:
  - \+ Write $y(x) = x^s \sum_{n=0}^{\infty} a_n x^n$ where $a_0 \neq 0$.
  - \+ Plug into the ODE. Renumber sums + gather terms of the same power. Note

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} (s+n) a_n x^{n+s-1} = \sum_{n=-1}^{\infty} (s+n+1) a_{n+1} x^{n+s}, \quad \frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} (s+n)(s+n-1) a_n x^{n+s-2} = \sum_{n=-2}^{\infty} (s+n+2)(s+n+1) a_{n+2} x^{n+s}$$

$$\text{ODE} = [\text{--}]\, x^s + [\text{--}]\, x^{s+1} + [\text{--}]\, x^{s+2} + \dots = 0$$

  - \+ For the ODE to vanish, the coefficients for <u>each power of x</u> <u>must vanish</u> <u>separately</u>.
  - \+ This gives a <u>recursion</u> (or <u>recurrence</u>) <u>relation</u> for the coefficients a_n in terms of the lower values of n (and allows you to solve for s)
- Example: Consider the familiar eqn $\frac{d^2y}{dx^2} - k^2 y = 0$ for k real.
  - \+ Set $y(x) = \sum_{n=0}^{\infty} a_n x^n$. The ODE is $\sum_{n=0}^{\infty} \left[n(n-1) a_n x^{n-2} - k^2 a_n x^n \right] = 0$
  - \+ We renumber the first sum, so the equation becomes

$$0 \cdot (-1) \cdot a_0 \cdot x^{-2} + 1 \cdot 0 \cdot a_1 \cdot x^{-1} + \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - k^2 a_n \right] x^n = 0$$

  - \+ The quantity in square brackets must vanish for <u>each power</u> of x. We find

$$a_{n+2} = \frac{k^2}{(n+2)(n+1)} a_n$$

  - \+ The solution of the recursion is $a_n = \frac{k^n}{n!} a_0$ for n even, $a_n = \frac{k^n}{n!} a_1$ for n odd
  - \+ a_0 + a_1 are integration constants. The series give solution

$$y(x) = a_0 \cosh(kx) + a_1 \sinh(kx)$$

— Partial Differential Equations and Separation of Variables

- Functions of several variables may solve equations involving partial derivatives. These are <u>partial</u> <u>differential</u> <u>equations</u> (PDEs).
  - + In general, these are even more complicated than ODEs
  - + Example: Laplace eqn $\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = 0$

- In some cases, we can break the PDE into one ODE per variable

  Method: + Write $\psi(x,y,z) = X(x)\,Y(y)\,Z(z)$ (for 3 variables)
  - + Plug in + divide equation by ψ.
  - + Then the equation splits into a term depending only on x, one on y, and one on z. These each must equal a constant.
  - + We get 3 eigenvalue equations

- Example: Take the Laplace equation with $\psi = X(x)Y(y)Z(z)$.
  - + Then

$$\nabla^2\psi = YZ\frac{\partial^2 X}{\partial x^2} + XZ\frac{\partial^2 Y}{\partial y^2} + XY\frac{\partial^2 Z}{\partial z^2} = 0$$

  - + After dividing

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = 0$$

  - + For this to be consistent everywhere, we <u>must</u> have

$$\frac{\partial^2 X}{\partial x^2} = \alpha_x X, \quad \frac{\partial^2 Y}{\partial y^2} = \alpha_y Y, \quad \frac{\partial^2 Z}{\partial z^2} = \alpha_z Z \quad \text{with } \alpha_x + \alpha_y + \alpha_z =$$

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