Second-Order Linear Equations

- The general form is $a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = f(x)$ (✱)
  - The associated homogeneous equation $a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = 0$ has two independent solutions $y_1(x)$ and $y_2(x)$. The complementary function is the linear combination $k_1y_1(x) + k_2y_2(x)$ for k_1, k_2 constant.
  - Then the general solution of (✱) is $y(x) = y_f(x) + k_1y_1(x) + k_2y_2(x)$ where $y_f(x)$ is some particular solution of (✱).
    + You can check that this $y(x)$ satisfies (✱) as long as $y_f(x)$ does.
    + k_1 and k_2 are chosen to satisfy initial or boundary conditions. We need two initial or boundary conditions to specify a solution completely for a 2nd-order equation.
  - If $y_1(x)$ solves (✱) for $f(x) = f_1(x)$ and $y_2(x)$ solves (✱) for $f(x) = f_2(x)$, then you can check that $y_1(x) + y_2(x)$ solves (✱) for $f(x) = f_1(x) + f_2(x)$

- Equations with constant coefficients
  - If the functions $a(x)$, $b(x)$, $c(x)$ are constants (✱) can be written
$$\frac{d^2y}{dx^2} + b_1\frac{dy}{dx} + b_2y = f(x)$$
  - First, the homogeneous case $f(x) = 0$
    + We want to be able to cancel d^2y/dx^2, dy/dx, and y against each other, so they need to be given by the same function
      - exponentials behave this way, so guess $y = e^{\lambda x}$
    + Then $y = e^{\lambda x}$, $\frac{dy}{dx} = \lambda e^{\lambda x}$, $\frac{d^2y}{dx^2} = \lambda^2 e^{\lambda x}$.
    The differential equation becomes
$$\frac{d^2y}{dx^2} + b_1\frac{dy}{dx} + b_2y = 0 \implies (\lambda^2 + b_1\lambda + b_2)e^{\lambda x} = 0$$
    + We have the auxiliary or characteristic equation
$$\lambda^2 + b_1\lambda + b_2 = 0$$

+ The auxiliary equation is quadratic for λ. In general, it has two distinct solutions, λ_1 + λ_2. Both $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ solve the homogeneous equation, so the complementary function is $k_1 e^{\lambda_1 x} + k_2 e^{\lambda_2 x}$

+ If $\lambda_1 = \lambda_2$ (the auxiliary eqn has a double root), we only get one solution. $y_1(x) = e^{\lambda_1 x}$. We can check that $y_2(x) = x e^{\lambda_1 x}$ is a solution.

Note: $dy_2/dx = e^{\lambda_1 x} + \lambda_1 x e^{\lambda_1 x}$, $\dfrac{d^2 y_2}{dx^2} = 2\lambda_1 e^{\lambda_1 x} + \lambda_1^2 x e^{\lambda_1 x}$, so

Also, the auxiliary equation is $(\lambda - \lambda_1)^2 = \lambda^2 - 2\lambda_1\lambda + \lambda_1^2 = \lambda^2 + b_1\lambda + b_2 = 0$

$\Rightarrow \lambda_1 = -b_1/2$, so $(\lambda + b_1/2) = 0$.

Compare to the DE: $(\lambda_1^2 + b_1\lambda_1 + b_2)\, x e^{\lambda_1 x} + (2\lambda_1 + b_1) e^{\lambda_1 x} = 0$

Both terms vanish

+ The auxiliary eqn can have complex solutions $\lambda_1 = \lambda_2^* = \alpha + ik$.

Then the complementary function can be written several ways

$$a_1 e^{(\alpha+ik)x} + a_2 e^{(\alpha-ik)x} = e^{\alpha x}\left(a_1' \sin(kx) + a_2' \cos(kx)\right)$$
$$= a' e^{\alpha x} \sin(kx + \delta) = a'' e^{\alpha x} \cos\left(kx + \delta - \tfrac{\pi}{2}\right)$$

- Physics examples

+ Harmonic Oscillator: this is a linear restoring force $-kx$, so

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0 \quad \text{with} \quad \omega_0^2 = k/m$$

Solutions are $x(t) = a_1' e^{i\omega_0 t} + a_2' e^{-i\omega_0 t} = a_1 \sin\omega_0 t + a_2 \cos\omega_0 t = a \sin(\omega_0 t + \delta)$

+ Damped harmonic oscillator: restoring force $-kx$ and linear drag force $-b'v$, so

$\dfrac{d^2x}{dt^2} + b\dfrac{dx}{dt} + \omega_0^2 x = 0$. Auxiliary eqn solutions $= -\dfrac{b}{2} \pm \dfrac{1}{2}\sqrt{b^2 - 4\omega_0^2}$

3 types of solutions are

Overdamped: $b > 2\omega_0$, so $x(t) = a_1 e^{-\left(\frac{b}{2} - \frac{1}{2}\sqrt{b^2-4\omega_0^2}\right)t} + a_2 e^{-\left(\frac{b}{2} + \frac{1}{2}\sqrt{b^2-4\omega_0^2}\right)t}$

Both are exponential decay b/c $b > 0$ and $b > \sqrt{b^2 - 4\omega_0^2}$

Underdamped: $b < 2\omega_0$, so solutions are complex exponentials

$$x(t) = a e^{-bt/2} \sin\left(\tfrac{1}{2}\sqrt{4\omega_0^2 - b^2}\, t + \delta\right)$$

Critically damped: $b = 2\omega_0$, double root

$$x(t) = (a_1 + a_2 t)\, e^{-bt/2}.$$

+ For Newton's eqn examples, like these, use $x(t=0) = x_0$ and $\dfrac{dx}{dt}(t=0) = v_0$ as usually init. cond.

- The inhomogeneous driven case $f(x) \neq 0$. All we need to find is a particular solution b/c we have seen how to find the complementary function. There are a few simple cases, then a general method.
  - If $f(x) = c_1 e^{\alpha x}$, guess particular solution $y(x) = A e^{\alpha x}$.
    Then
    $$\frac{d^2y}{dx^2} + b_1 \frac{dy}{dx} + b_2 y = f(x) \Rightarrow A(\alpha^2 + b_1\alpha + b_2) e^{\alpha x} = c_0 e^{\alpha x}$$
    The D.E. determines the coefficient $A = \frac{c_1}{\alpha^2+b_1\alpha+b_2} = c_0/(\alpha-\lambda_1)(\alpha-\lambda_2)$
  - Similarly, if $f(x) = c_0 e^{i\omega x}$, guess $y(x) = A e^{i\omega x}$. Then
    $$A = \frac{c_0}{(-\omega^2 + i\omega b_1 + b_2)}$$
    . This is similar to the real exponential
  - If the driving term $e^{\alpha x}$ or $e^{i\omega x}$ has $\alpha = \lambda_1$ or λ_2, $e^{\alpha x}$ is the complementary solution. $y(x) = Axe^{\alpha x}$ works out to the particular solution. If $\alpha = \lambda_1 = \lambda_2$, we need $y(x) = Ax^2 e^{\alpha x}$. Check these!
  - These are examples of the <u>method of undetermined coefficients</u>. In general, if $f(x) = (c_0 + c_1 x + \cdots + c_n x^n) e^{\alpha x}$ (α complex), the particular solution is $y(x) = (a_0 + a_1 x + \cdots + a_n x^n) e^{\alpha x}$
    (If $\alpha = \lambda_1$ and/or λ_2, try $x(a_0 + \cdots a_n x^n) e^{\alpha x}$ or $x^2(a_0 + \cdots + a_n x^n) e^{\alpha x}$)
    The differential equation gives a series of eqns. for $a_0, a_1, \cdots a_n$
  - If $f(x)$ is a periodic function, write it as a Fourier series.
    Then $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n x}$ and the particular solution is the sum of particular solutions for each term $y(x) = \sum_n \frac{c_n}{(-\omega_n^2 + i\omega_n b_1 + b_2)} e^{i\omega_n x}$
  - Similarly, we can write more general $f(x)$ as a Fourier integral
    $$f(x) = \int d\omega \frac{1}{\sqrt{2\pi}} \tilde{f}(\omega) e^{i\omega x}, \text{ so } y(x) = \int d\omega \frac{1}{\sqrt{2\pi}} \frac{\tilde{f}(\omega)}{(-\omega^2 + i\omega b_1 + b_2)} e^{i\omega x}$$

- Examples:
  - Ballistic motion with air resistance $\vec{F}/m = -b\vec{v} - g\hat{y}$.
    Then
    $$\frac{d^2x}{dt^2} + b\frac{dx}{dt} = 0, \quad \frac{d^2y}{dt^2} + b\frac{dy}{dt} = -g$$

The complementary functions are $x(t) = x_0 + x_1 e^{-bt}$, $y(t) = y_0 + y_1 e^{-bt}$

The particular solution for $y(t)$ should be given by polynomial $a_0 + a_1 t$. Checking, we find $ba_1 = -g$. Altogether, the general solutions are

$$x(t) = x_0 + x_1 e^{-bt}, \quad y(t) = y_0 - \frac{g}{b}t + y_1 e^{-bt}$$

If initial conditions are $x(0) = 0$, $y(0) = 0$, $v_x(0) = V_0 \cos\theta$, $v_y(0) = V_0 \sin\theta$,

$$x(t) = \frac{V_0 \cos\theta}{b}\left(1 - e^{-bt}\right), \quad y(t) = \frac{V_0 \sin\theta + g/b}{b}\left(1 - e^{-bt}\right) - \frac{g}{b}t$$

$$= \frac{V_0 \sin\theta + g/b}{V_0 \cos\theta}\, x(t) + \frac{g}{b^2}\ln\left(1 - \frac{bx}{V_0 \cos\theta}\right)$$

+ Forced oscillator

This is
$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + \omega_0^2 x = F_0 e^{i\omega t}$$

for a periodic driving force (sine or cosine behavior from 2 terms).

The particular solution is
$$x(t) = \frac{F_0}{\omega_0^2 - \omega^2 + ib\omega} = \frac{[\omega_0^2 - \omega^2 - ib\omega]F_0}{(\omega_0^2 - \omega^2)^2 + b^2\omega^2}$$

The complementary function always dies off exponentially, so they are <u>transients</u> while this is the <u>steady state solution</u>

Note that the solution is maximized when $\omega = \omega_0$ (<u>resonance</u>).