First-Order ODEs

- A general first order ODE can be written in the form

$$\frac{dy}{dx} = G(x,y) \equiv -\frac{P(x,y)}{Q(x,y)} \qquad (*)$$

We'll look at some special nonlinear cases first, then focus on linear eqns

- Separable Variables
  - Take our general form in the case $P = P(x)$, $Q = Q(y)$
  - The solution can be written $\int dy\, Q(y) = -\int dx\, P(x) + C$
    where C is a constant of integration (explicit, so we don't forget)
  - Example: vertical motion with turbulent air resistance
    - Newton's law becomes $\frac{dv}{dt} = g - bv^2$ for purely downward motion.
    - There is a terminal velocity $\frac{dv}{dt} = 0$ for $v = v_t = \sqrt{g/b}$.
    - The solution is given by

$$\int \frac{dv}{(v_0^2 - v^2)} = b\int dx + C_0 = \frac{1}{2v_0}\int dv\left(\frac{1}{v_0+v} + \frac{1}{v_0-v}\right)$$

or

$$\ln\left(\frac{v_0+v}{v_0-v}\right) = 2v_0 bt + C_1$$

    - If $v(t=0) = 0$, $C_0 = C_1 = 0$. and $\frac{v_0+v}{v_0-v} = e^{2\sqrt{bg}\,t} \Rightarrow v = v_0 \tanh(\sqrt{bg}\,t)$

- Exact equations
  - Rewrite $(*)$ as $Q\,dy + P\,dx = 0$
    - This is <u>exact</u> if it is the differential of some function F

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy \quad \text{or} \quad P = \frac{\partial F}{\partial x},\ Q = \frac{\partial F}{\partial y}$$

    - Because partial derivatives commute, an equation is exact if

$$\frac{\partial^2 F}{\partial x \partial y} = \boxed{\frac{\partial Q}{\partial x}} = \frac{\partial^2 F}{\partial y \partial x} = \boxed{\frac{\partial P}{\partial y}}$$

  - So if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$,

$$F(x,y) = \int dx\, P(x,y) + f(y)$$

with $f(y)$ determined by setting $\frac{\partial F}{\partial y} = Q$.

- Then $F(x,y) = \text{constant}$. The solution follows by solving for $y(x)$ and then fixing the constant using the initial condition
- Work an example:

$$xy' + 3x + y = 0 \Rightarrow (3x+y)dx + x\,dy = 0$$

  + Check $P = 3x+y$, $Q = x$, $\partial P/\partial y = 1 = \partial Q/\partial x$
  + Then $F(x,y) = \int dx\, P + f(y) = \frac{3x^2}{2} + xy + f(y)$

    and $\partial F/\partial y = x + \partial f/\partial y = Q = x \Rightarrow f = \text{const}$
  + Our solution is $F = 3x^2/2 + xy = c \Rightarrow y = \frac{c}{x} - \frac{3x}{2}$

- Linear First-Order Equations
  - A general first-order linear equation is $dy/dx + P(x)y = Q(x)$ (*)
    + If $Q=0$, this is <u>homogeneous</u>; $Q \neq 0$ is a <u>source</u> or <u>driving</u> term.
    + If $y = y_Q$ is any solution to (*), the general solution is

      $y = y_Q + c y_h$ where y_h solves the homogeneous equation $dy/dx + P(x)y = 0$
      $c = \text{const}$
  - So we want to solve the homogeneous equation.
    + It is separable: $dy/y = -P(x)\,dx$
    + The solution is $y = (\text{const})\, e^{-I(x)}$ where $I(x) = \int dx\, P(x)$
  - Now we need <u>any</u> solution to (*).
    + Note that $dI/dx = P$. Therefore $\frac{d}{dx}(e^I y) = e^I\left(\frac{dy}{dx} + \frac{dI}{dx} y\right)$

      $= e^I\left(\frac{dy}{dx} + P(x)y\right)$
    + That means (*) is

$$Q = y' + P(x)y = e^{-I}\frac{d}{dx}(e^I y) \Rightarrow y_Q = e^{-I}\int dx\, e^I Q$$

  - Example: We have a sample of radium (with N_0) atoms at time t_0. Radium decays to radon with lifetime τ_1. Radon decays with lifetime τ_2. How many radon atoms are there at time t if we start with none?

+ First, we set up the differential equations. Let $N(t) = \#$ radium and $n(t) = \#$ radon.

$$\frac{dN}{dt} = \frac{-N}{\tau_1} = -\left(\frac{\#\text{ lost per}}{\text{time}}\right), \quad \frac{dn}{dt} = \frac{-n}{\tau_2} + \frac{N}{\tau_1} = -\left(\frac{\#\text{ decay}}{\text{time}}\right) + \left(\frac{\#\text{ gained}}{\text{time}}\right)$$

+ The equation for N is homogeneous:

$$N(t) = (\text{const}) \exp\left[\int \left(\frac{-1}{\tau_1}\right) dt\right] = N_0 e^{-t/\tau_1} \text{ with initial conditions}$$

That means

$$\frac{dn}{dt} + \frac{n}{\tau_2} = \frac{N_0}{\tau_1} e^{-t/\tau_1} \text{ We can try 2 methods}$$

+ First, as above, $I = t/\tau_2$, so

$$n_Q = e^{-t/\tau_2} \int dt \left[\frac{N_0}{\tau_1} e^{t\left(\frac{1}{\tau_2} - \frac{1}{\tau_1}\right)}\right] = \frac{N_0}{\tau_1} \frac{1}{\frac{1}{\tau_2} - \frac{1}{\tau_1}} e^{-t/\tau_1}$$

The general solution is

$$n(t) = A e^{-t/\tau_2} + \frac{N_0 \tau_2}{\tau_1 - \tau_2} e^{-t/\tau_1} = \frac{N_0 \tau_2}{\tau_1 - \tau_2}\left(e^{-t/\tau_1} - e^{-t/\tau_2}\right),$$

for $n(0) = 0$.

+ Alternative solution: Guess a solution $n(t) = Ce^{-t/\tau_1}$. Then

$$-\frac{C}{\tau_1} + \frac{C}{\tau_2} = \frac{N_0}{\tau_1} \Rightarrow n_Q(t) = \frac{N_0}{\tau_1} \frac{1}{1/\tau_2 - 1/\tau_1} e^{-t/\tau_1} \text{ as above}$$