Differential Equations

Classification:

- Differential equations relate a function y (or f) of one or more independent variables $x, t, \ldots$ to its derivatives and specified functions of the independent variables.
  There are very many important physics examples:
  - Laplace equation $\nabla^2 \psi(\vec{x}) = 0$ in E+M, gravity, thermal physics
  - Poisson equation $\nabla^2 \psi(\vec{x}) = -\rho(\vec{x})$ like Laplace but with a source (such as charge in EM)
  - Diffusion equation $D \frac{\partial f}{\partial t} = \nabla^2 f$
  - Wave equation $\frac{1}{c^2} \frac{\partial^2 f}{\partial t^2} = \nabla^2 f$ $\leftarrow$ many kinds of waves
  - The F.T. in time of diffusion + wave equations is the Helmholtz equation $\nabla^2 f \pm k^2 f = 0$
  - Schrödinger equations $-\frac{\hbar^2}{2m} \nabla^2 \psi + V(\vec{x}) \psi = i\hbar \frac{\partial \psi}{\partial t}$ or $= E \psi(\vec{x})$
  - And many more. Newton's law is one of the most familiar: $m \frac{d^2 \vec{x}}{dt^2} = \vec{F}(\vec{x}, d\vec{x}/dt, t)$

- Diff. Eqs. are classified by the # of independent variable
  - Partial Diff. Eqs. (PDEs) involve multiple independent variables + partial derivatives
    - 3D Laplace equation $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f(x,y,z) = 0$
    - 1+1 D wave equation $\frac{1}{c^2} \frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2} = 0$
  - Ordinary Diff. Eqs. (ODEs) are for functions of a single variable
    - Newton's law
    - 1D Helmholtz eqn $\frac{d^2 f}{dx^2} + k^2 f = 0$

- The <u>order</u> of a DE is the highest derivative in the equation.
  - We are most concerned with <u>first-order</u> equations (involving dy/dx, $y(x)$, and functions of x) in the ODE case) and <u>2^{nd} order</u> equations (involving second derivatives)
  - Second order equations are the most prominent in physics, but first-order equations are very important in intermediate steps.
- <u>Linear</u> differential equations have only 1 power of y or its derivatives in any term, ie, no y^2 or $y\frac{dy}{dx}$ type terms
  - We will mostly focus on linear DEs as nonlinear equations are often much harder to solve
  - However, nonlinear DEs are very common, and solving them is often cutting-edge science + computing.
  - Note that the independent variable can be involved nonlinearly in a linear DE. That is, $\frac{dy}{dx} + x^2 y = x^3$ is linear
- The general solution has constants of integration (usually) of number = order of DE. These are fixed by specifying initial conditions (in typically time) or boundary conditions (in space)
- Two simple examples
  - Solve $d^2y/dx^2 = \alpha^2 y$ for $\alpha > 0$
    + You can check that $y = Ae^{\alpha x} + Be^{-\alpha x}$
    + If we insist $y(0) = 0$ and $y(\frac{\ln 2}{\alpha}) = \frac{3}{4}$ (boundary conditions), we find $A = -B$ and $\frac{3}{4} = A(2 - \frac{1}{2}) \Rightarrow A = \frac{1}{2} \Rightarrow y(x) = \sinh(\alpha x)$
  - Newton's law for vertical motion
    + Start with $d^2y/dt^2 = -g \Rightarrow \frac{dy}{dt} = v_0 - gt \Rightarrow y = y_0 + v_0 t - \frac{1}{2}gt^2$
    + y_0 and v_0 are determined by initial conditions at $t=0$.