- Linear Operators: A linear operator A is a function that takes vectors from vector space X into vectors from vector space Y, ie, $|y\rangle = A|x\rangle$ such that $A(\lambda|a\rangle + \rho|b\rangle) = \lambda A|a\rangle + \rho A|b\rangle$
  - Properties
    + If $A: X \to Y$ and $B: X \to Y$, $(A+B)|x\rangle = A|x\rangle + B|x\rangle$
    + For scalar λ, $(\lambda A)|x\rangle = \lambda(A|x\rangle)$ (ie, operators can be multiplied by scalars)
    + For $A: Y \to Z$, $B: X \to Y$, then $(AB)|x\rangle = A(B|x\rangle)$. This is composition of operators (associativity)
    + Note: Generally $AB|x\rangle \neq BA|x\rangle$. In fact, $A|x\rangle$ may not even be in the vector space B acts on! Operator action is not commutative in general
    + Two operators $A: X \to Y$, $B: X \to Y$ are equal if $A|x\rangle = B|x\rangle$ for all vectors $|x\rangle$. But linearity means this is true if $A|e_i\rangle = B|e_i\rangle$ for all basis vectors $|e_i\rangle$ in any basis. (Why?)
    + There is a null (zero) operator O s.t. $O|x\rangle = \emptyset$ for all $|x\rangle$ and an identity operator $I: X \to X$ s.t. $I|x\rangle = |x\rangle$ for all $|x\rangle$
    + If $A: X \to Y$ and X, Y have the same dimensionality, A may have an inverse operator A^{-1} s.t. $A^{-1}A = I_X$, $AA^{-1} = I_Y$. Operators that do not have an inverse are singular.
  - Operators also have elements in terms of basis sets.
    + Suppose $A: X \to Y$ and X has basis $\{|e_1\rangle, \ldots |e_N\rangle\}$ and Y has $\{|f_1\rangle, \ldots |f_M\rangle\}$.
    + Then $A|e_j\rangle$ is a vector in Y and can be written

$$A|e_j\rangle = A_{i,j}|f_i\rangle \quad \text{(Note sum over } i\text{)}.$$

The $A_{i,j}$ scalars are the elements of A for these bases. (Usually drop the comma).

+ We can now write any operator action on components.

If $|y\rangle = A|x\rangle$, then

$$|y\rangle = A(x_j |e_j\rangle) = x_j A_{ij} |f_i\rangle \equiv y_i |f_i\rangle, \text{ so } y_i = A_{ij} x_j$$

+ Note that $A_{ij} = \langle f_i | A | e_j \rangle$

- Eigenvectors + eigenvalues
  + Consider an operator $A: X \to X$. Usually $|x\rangle$ and $A|x\rangle$ are linearly independent (for some given vector $|x\rangle$). + But there are special vectors for our given operator s.t. $A|x\rangle = \lambda |x\rangle$ where λ is a scalar. These special vectors are eigenvectors of A, and the scalar values λ are called eigenvalues.
  + For an N-dim space, A generally has N eigenvectors $|x_i\rangle$ and N corresponding eigenvalues λ_i. The eigenvalues may be degenerate (ie, some of them might be the same).
  + Finding eigenvectors + eigenvalues is important in many physics applications. We will study (a) when eigenvectors form a basis and (b) how to find them.
  + Note: if $|x\rangle$ is an eigenvector of A, $c|x\rangle$ is also for any scalar c because $A(c|x\rangle) = cA|x\rangle = c(\lambda|x\rangle) = \lambda(c|x\rangle)$. It has the same eigenvalue λ as $|x\rangle$ does.