- <u>Inner Product</u>: generalization of the dot product
  - An inner product is a function of 2 vectors that gives a scalar
  - So for 2 vectors $|a\rangle, |b\rangle$, the inner product $\langle a|b\rangle$ is a real number (for a real vector space) or a complex number (complex vector space)
  - The inner product satisfies

$$\langle a|(\lambda|b\rangle + \rho|c\rangle) = \lambda\langle a|b\rangle + \rho\langle a|c\rangle$$

$$\langle a|b\rangle = \langle b|a\rangle^* \quad (= \langle b|a\rangle \text{ for a real space})$$

(Note that the 1st and 2nd vector positions are <u>not</u> equivalent)

and $\langle a|a\rangle \geq 0$ ie, it is <u>positive semi-definite</u>

  - The <u>norm</u> of a vector is $||a\rangle| \equiv \sqrt{\langle a|a\rangle}$ and $||a\rangle| = 0 \Rightarrow |a\rangle = \emptyset$
  - How does the usual dot product fit in?
  - In physics, especially relativity, we sometimes define scalar products that are <u>not</u> positive semi-definite

- In terms of components, the usual dot product is

$$\vec{a}\cdot\vec{b} = a_i b_i = \delta_{ij} a_i b_j \quad \text{(real space)}$$

  + A general inner product can be written in components as

$$\langle a|b\rangle = G_{ij} a_i^* b_j \quad \text{with } G_{ij} = G_{ji}^*$$

  + G_{ij} is a set of numbers that make up a <u>metric</u>. There are $N(N+1)/2$ independent scalars in a real metric and $N(N-1)/2$ complex + N real scalars in a complex metric
  + How can we see this? Well, in a given basis

$$\langle a|b\rangle = \langle a|(b_j|e_j\rangle) = b_j\langle a|e_j\rangle = b_j\cdot\langle e_j|a\rangle^*$$
$$= b_j\left[\langle e_j|(a_i|e_i\rangle)\right]^* = a_i^* b_j \langle e_j|e_i\rangle^* = \langle e_i|e_j\rangle a_i^* b_j$$

  The metric is $G_{ij} = \langle e_i|e_j\rangle$.

- We usually want to work with an <u>orthonormal basis</u>.
  + This is when each basis vector has unit norm (is <u>normalized</u>), and they are all orthogonal to each other,
  + This means that the metric $G_{ij} = \langle e_i|e_j\rangle = \delta_{ij}$
  + It's especially useful for finding components: In an orthonormal basis,

$$\langle e_i|a\rangle = \langle e_i|(a_j|e_j\rangle) = a_j\langle e_i|e_j\rangle = \delta_{ij} a_j = a_i$$

  The <u>component</u> is <u>the inner product</u> with the <u>basis vector</u>
  So, you can either work out the linear combination or just do inner products.
  + For an orthonormal basis, the norm is

$$\||a\rangle\|^2 = \langle a|a\rangle = \delta_{ij} a_i^* a_j = \sum_{i=1}^{N} |a_i|^2$$

  This leads directly to <u>Bessel's inequality</u> $\langle a|a\rangle \geq \sum_{i=1}^{K} |a_i|^2$ for $K \leq N$.

- Inner products + norms satisfy (proofs in reading)
  + (Cauchy-) Schwarz inequality $|\langle a|b\rangle|^2 \leq \||a\rangle\|\,\||b\rangle\|$
  + Triangle inequality $\||a\rangle + |b\rangle\| \leq \||a\rangle\| + \||b\rangle\|$

- How to turn a general basis into an orthonormal basis: Gram-Schmidt procedure
  - + We have a basis $\{|f_i\rangle\}$ and want to convert it to a new orthonormal basis $\{e_i\}$
  - + Define $|e_1\rangle = |f_1\rangle / ||f_1\rangle|$: Check normalization.
  - + Then define $|e_2'\rangle = |f_2\rangle - (\langle e_1|f_2\rangle)|e_1\rangle$ and $|e_2\rangle = |e_2'\rangle / ||e_2'\rangle|$.
    Check that $\langle e_1|e_2\rangle = 0$ and $||e_2\rangle| = 1$
  - + Then $|e_3'\rangle = |f_3\rangle - (\langle e_1|f_3\rangle)|e_1\rangle - (\langle e_2|f_3\rangle)|e_2\rangle$
    and $|e_3\rangle = |e_3'\rangle / ||e_3'\rangle|$. Check.
  - + Keep repeating until you have completed the basis