concepts shortly

- **Mathematical Definition of vectors**

A more abstract definition will be useful for differential equations, data analysis, and quantum mechanics

- Think of vectors as mathematical objects. Can be denoted with bold, an arrow $\vec{x}$, sometimes an underline $\underline{x}$, just the components x_i, or as a <u>ket</u> $|x\rangle$. We will use $\vec{x}$ for vectors like position, momentum, etc; index notation x_i for specific components; kets $|x\rangle$ for general vectors

- A <u>vector space</u> is a set of <u>vectors</u> $\{|a\rangle, |b\rangle, \ldots\}$ associated with a set of <u>scalars</u> (numbers) such that
  - There is vector addition $|a\rangle + |b\rangle$ where
    + the vector space is <u>closed under addition</u> ($|a\rangle + |b\rangle$ is a vector)
    + Addition is <u>commutative</u> $|a\rangle + |b\rangle = |b\rangle + |a\rangle$
    + Addition is <u>associative</u> $(|a\rangle + |b\rangle) + |c\rangle = |a\rangle + (|b\rangle + |c\rangle)$
  - Vectors can be multiplied by scalars $\lambda|a\rangle$ where
    + the vector space is <u>closed under scalar multiplication</u> ($\lambda|a\rangle$ is a vector)
    + Scalar multiplication is <u>associative</u> $\lambda(\rho|a\rangle) = (\lambda\rho)|a\rangle$
    + scalar multiplication is <u>distributive</u> (both ways)
      $(\lambda + \rho)|a\rangle = \lambda|a\rangle + \rho|a\rangle$ and $\lambda(|a\rangle + |b\rangle) = \lambda|a\rangle + \lambda|b\rangle$

- There is a zero vector $\emptyset$ (written w/o the ket) such that $|a\rangle + \emptyset = |a\rangle$ for any vectors $|a\rangle$ (Note: $\emptyset = 0 \cdot |a\rangle$ for any $|a\rangle$)
- Multiplication by 1 is the identity: $\underline{1} \cdot |a\rangle = |a\rangle$ for any vector $|a\rangle$
- Multiplication by -1 gives a negative vector $(-1)|a\rangle \equiv -|a\rangle$ such that $|a\rangle + (-|a\rangle) = \emptyset$ (vector subtraction)

- Note: the scalars can be real numbers $\Rightarrow$ real vector space, complex numbers $\Rightarrow$ complex vector space, ...
  We will work with both cases

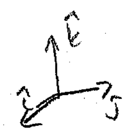
- Give some examples of vector spaces (esp columns)

- Basis: like choosing axes
  - Suppose you consider $\alpha|a\rangle$ for any $\alpha \in \mathbb{R}$ in a real vector space. This draws a line in the vector space (in a complex space for $\alpha \in \mathbb{C}$, you get a whole complex plane)
  - Similarly, $\alpha_1|a_1\rangle + \alpha_2|a_2\rangle + \cdots + \alpha_n|a_n\rangle$ sweeps out a (real or complex) space. This space is the span of $|a_1\rangle, \ldots |a_n\rangle$

draw
    + Suppose $\alpha_1|a_1\rangle + \cdots + \alpha_n|a_n\rangle = \emptyset$ at some value other than $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. Then $|a_n\rangle$ is a linear combination of the others. Then $|a_1\rangle, \ldots, |a_n\rangle$ are linearly dependent. The span is less than n-dimensional
    + If $\alpha_1|a_1\rangle + \cdots + \alpha_n|a_n\rangle = \emptyset$ only for $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$, the vectors are linearly independent. They span $\mathbb{R}^n$ or $\mathbb{C}^n$ ~ n-dim space
  - An N-dim. vector space is one in which there are sets of N vectors that are linearly independent but also in which all sets of $N+1$ vectors are linearly dependent. Essentially, there are no more new directions.

    Note: It is possible to have infinite dimensional vector spaces, but we will mostly just talk about finite dimensions

- A **basis** for an N-dim space is a set $\{|e_1\rangle, \ldots |e_N\rangle\}$ of linearly independent vectors
  - + Any vector can be written in a unique way as a linear combination of basis vectors
    $$|x\rangle = x_1|e_1\rangle + x_2|e_2\rangle + \cdots + x_n|e_n\rangle = x_i|e_i\rangle \text{ (why?)}$$
  - + The coefficients x_i are the components of $|x\rangle$ with respect to that basis
  - + A given vector space has (many) different basis sets.
    We could have, for example, another basis $\{|e_{1'}\rangle, \ldots |e_{N'}\rangle\}$
    Then our same vector $|x\rangle$ has different components in this basis
    $$|x\rangle = x_{1'}|e_{1'}\rangle + \cdots x_{N'}|e_{N'}\rangle = x_{i'}|e_{i'}\rangle$$
    I put the prime on the index b/c it's the basis that's different, Not the vector
- Look at examples



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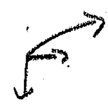
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