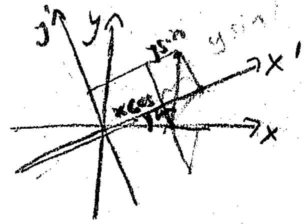
— Change of basis

- Let's remember that the <u>same</u> vector $|x\rangle$ has <u>different</u> components when written w.r.t. different basis sets
$$|x\rangle = x_i|e_i\rangle = x_j'|e_j'\rangle \quad \text{where } x_i \neq x_i'$$
  + This means we write $|x\rangle$ as different columns when translating to matrix form
$$|x\rangle \xrightarrow[|e\rangle \text{ basis}]{} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \equiv x \quad \text{vs} \quad |x\rangle_{\overrightarrow{|e'\rangle}} \begin{bmatrix} x_1' \\ \vdots \\ x_N' \end{bmatrix} \equiv x'$$
  + In general, each $|e_i'\rangle$ can be written as a linear combination of the $|e_i\rangle$, so $|e_j'\rangle = |e_i\rangle S_{ij}$.
    That means $x_j'|e_j'\rangle = (S_{ij}x_j')|e_i\rangle = x_i|e_i\rangle$,
    That is the component version of matrix multiplication
$$x' = S^{-1}x. \quad S = \underline{\text{transformation matrix}}$$
  + But remember that the different columns x + x' represent the same vector from different points of view
- Suppose that the original basis is orthonormal
  + Then the inner product is $\langle y|x\rangle = y^\dagger x$.
    Using the change of basis, $\langle y|x\rangle = (Sy')^\dagger(Sx') = y'^\dagger(S^\dagger S)x'$
    This is an inner product with metric $G = S^\dagger S$.
  + From the basis vector point of view, we remember that
$$G_{ij} = \langle e_i'|e_j'\rangle = S_{ki}^* \langle e_k|e_l\rangle S_{lj} = S_{ki}^* \delta_{kl} S_{lj} = (S^\dagger)_{ik}S_{kj}$$
  + <u>If the transformation matrix is</u> <u>unitary</u> $S^\dagger = S^{-1}$, the <u>new</u> basis <u>is</u> <u>still</u> <u>orthonormal</u>.

+ For a real vector space, orthogonal transformations leave the basis orthonormal
+ When the new basis is orthonormal, the new components are $(x')_i = \langle e'_i | x \rangle = e_i^\dagger x$

- Rotation matrices are orthogonal matrices.
  + Consider position vectors $\vec{x}$. We typically measure coordinates/components w.r.t. orthonormal basis vectors $\hat{k}, \hat{i}, \hat{j}$.
  + What if we want to measure w.r.t. a rotated axis?
  $$\vec{x} = x_1\hat{i} + x_2\hat{j} + x_3\hat{k} \quad \text{vs} \quad \vec{x} = x_1'\hat{i}' + x_2'\hat{j}' + x_3'\hat{k}' ?$$
  + The change of basis should keep lengths and angles the same. Therefore, we should always be able to write dot products as $y_i x_i = y_j' x_j'$ (or $y^T x = y'^T x'$ in matrix form)
  + That means a rotation is an orthogonal change of basis
  $$x' = Rx \quad \text{for } R^T R = 1.$$
  + Example: A rotation around the z axis takes the form
  $$R_z(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ from  etc
  + Any 3D rotation can be described by 3 rotations by designated Euler angles: first, rotate by α around z axis, then by β around the new y axis, then by γ around the new z axis
  $$R = R_z(\gamma) R_y(\beta) R_z(\alpha)$$
  + Rotation matrices satisfy $\det(R) = +1$. Orthogonal matrices with $\det(R) = -1$ represent a rotation combined with a refletion through the origin given by multiplication with (-1).

- Similarity transformations: Remember that linear operator action $|y\rangle = A|x\rangle$ converts to matrix notation in the $\{|e_i\rangle\}$ basis as $y = Ax$. In the $\{|e_j'\rangle\}$ basis, it is $y' = A'x'$. We know $y = Sy'$ and $x = Sx'$, so $Sy' = ASx' \Rightarrow A' = S^{-1}AS$. This is a similarity transformation for the elements of an operator from one basis to another.

- Diagonalization?
  - When can we find a basis such that the matrix representation of a linear operator A is diagonal?
    + Suppose the eigenvectors of A form a basis $\{|e_i'\rangle\}$. Then
      $$A|e_1'\rangle = \lambda_1 |e_1'\rangle, \quad A|e_2'\rangle = \lambda_2 |e_2'\rangle, \text{ etc.}$$
    + The matrix elements of a linear operator are given by
      $$A|e_j'\rangle = \sum_i A'_{ij} |e_i'\rangle \implies A'_{ij} = \lambda_i \delta_{ij} = \text{diagonal matrix!}$$
    + A matrix is <u>diagonalizable</u> when its eigenvectors form a basis.
    + If we start in <u>any</u> other basis, the eigenvectors $|e_i'\rangle = X_{(i)j}|e_j\rangle$, etc are written as columns $X_{(1)}, X_{(2)}$, etc. Then <u>build</u> a matrix S with columns $X_{(i)}$.
      $$S = [X_{(1)} | X_{(2)} | \cdots | X_{(N)}]$$
      By matrix multiplication $AS = [\lambda_1 X_{(1)} | \cdots | \lambda_N X_{(N)}]$

      But $S^{-1}X_{(1)} = \begin{bmatrix} 1 \\ 0 \\ \vdots \end{bmatrix}$, etc, so $S^{-1}AS = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \end{bmatrix} = A'$
    + The matrix S built by putting the eigenvectors of A as columns is the similarity transformation that diagonalizes A (if the eigenvectors make a basis)
  - The eigenvectors of a normal matrix are orthonormal
    + Therefore, if we take $U = [X_{(1)} | X_{(2)} | \cdots]$, then $U^\dagger = \begin{bmatrix} X_{(1)}^\dagger \\ \hline X_{(2)}^\dagger \\ \vdots \end{bmatrix}$,

      we find $(U^\dagger U)_{ij} = X_{(i)}^\dagger X_{(j)} = \delta_{ij} \implies U^\dagger = U^{-1}$
    + Normal matrices are diagonalized by <u>unitary</u> transformations
    + Similarly, real symmetric matrices are diagonalized by orthogonal transformations
    + Logic like the above shows that the columns of any unitary matrix are orthonormal (same for real orthogonal matrices)
  - The trace of a diagonalizable matrix = sum of eigenvalues (in any basis)

    The determinant of a diagonalizable matrix = product of eigenvalues