- Eigenvectors and Eigenvalues of a Square Matrix
  - Remember that a linear operator usually changes the direction of a vector, even when it gives a vector in the same vector space.
    - Given an operator $A: X \to X$, a vector $|x\rangle$ is an <u>eigenvector</u> of A when $A|x\rangle$ is parallel to $|x\rangle$, ie $A|x\rangle = \lambda|x\rangle$, where λ is a scalar number called the <u>eigenvalue</u>
    - This type of operator becomes a square matrix in matrix notation. So the eigenvalue equation is $Ax = \lambda x$ where x is a column vector <u>NOT equal to the zero vector</u>
    - Generally, an $N \times N$ matrix A has N eigenvalues* $\lambda_1, \lambda_2, \ldots \lambda_N$ and N corresponding eigenvectors. If all the eigenvalues are different, the eigenvectors form a basis. However, if some of the eigenvalues are the same (<u>degenerate</u>), the eigenvectors might not make a basis. *For complex vector spaces. Perhaps not for real ones.
  - Eigenvectors + eigenvalues have many physical applications. How do we find them?
    - The defining equation to solve is $Ax = \lambda x$ where both the eigenvalue λ and eigenvector x are unknown
    - If we rewrite this equation as $(\lambda \mathbb{1}_N - A)x = 0$, we see that there is only a nonzero solution for x if $(\lambda \mathbb{1}_N - A)$ does <u>not</u> have an inverse matrix.
    - That means $\det(\lambda \mathbb{1} - A) = 0$, <u>the characteristic equation</u>. The det. is an order N polynomial in λ, so you must find the roots $\lambda_1, \lambda_2, \ldots \lambda_N$. That gives you the eigenvalues.
    - Note: for a real vector space, there may not be N real solutions to the characteristic equation.

+ For each eigenvalue λ_i, solve $Ax = \lambda_i x$ to find the eigenvectors.

Note: If x is an eigenvector with e'value λ_i, so is any multiple cx (for c a scalar) b/c $A(cx) = c(Ax) = c(\lambda_i x) = \lambda_i (cx)$. So you are free to fix the normalization of x.

+ If an eigenvalue is degenerate, $Ax = \lambda_i x$ generally has multiple distinct (linearly indep.) solutions (one per multiplicity of λ_i). If there are not that many indep. e'vectors, A is defective.

+ Work examples.

- When are eigenvectors guaranteed to form an orthonormal basis? For certain special matrices. (starting in a different orthonormal basis)

+ Diagonal Matrices have all elements zero except for the main diagonal $A_{11}, A_{22}, \ldots$. What are det, eigenvalues, eigenvectors?

+ In real spaces, symmetric matrices $A^T = A$ have an orthonormal basis of eigenvectors. Real antisymmetric matrices $A^T = -A$ do in complex space.

+ In complex space, ~~real~~ orthogonal matrices $A^T = A^{-1}$. These leave the inner product of real vectors unchanged since

$$\langle (Ay) | (Ax) \rangle = (Ay)^T (Ax) = y^T A^T A x = y^T x = \langle y | x \rangle.$$

Also, can you show $\det(A) = \pm 1$? Again, eigenvectors are orthonormal basis in complex space, not in real space.

+ In complex vector spaces, the corresponding types of matrices are Hermitian/anti-Hermitian matrices $A^\dagger = \pm A$ and unitary matrices $A^\dagger = A^{-1}$.

+ The most general type of matrix with eigenvectors forming an orthonormal basis is normal matrices, which follow $A^\dagger A = A A^\dagger$.

- Why do the eigenvectors of a normal matrix form an orthonormal basis? + For normal A with some eigenvalue λ and corresponding e'vector x, define $B = \lambda \mathbb{1}_N - A$, so $Bx = \emptyset$. We note 2 things:

a) $B^\dagger B = |\lambda|^2 \mathbb{1} - \lambda A^\dagger - \lambda^* A - A^\dagger A = |\lambda|^2 \mathbb{1} - \lambda A^\dagger - \lambda^* A - A A^\dagger = B B^\dagger$

so b) $0 = (Bx)^\dagger (Bx) = x^\dagger B^\dagger B x = x^\dagger B B^\dagger x = (B^\dagger x)^\dagger (B^\dagger x) \Rightarrow B^\dagger x = \emptyset$.

$\Rightarrow A^\dagger x = \lambda^* x$, so the e'values of $A^\dagger$ are the conjugates of those for A with the same eigenvectors.

+ Now consider 2 eigenvectors $x_{(i)}$ and $x_{(j)}$ with eigenvalues λ_i and λ_j.

We have $x_{(j)}^\dagger A x_{(i)} = x_{(j)}^\dagger(\lambda_i x_{(i)}) = \lambda_i (x_{(j)}^\dagger x_{(i)})$. But also $x_{(j)}^\dagger A x_{(i)} = (A^\dagger x_{(j)})^\dagger x_{(i)}$

By the above, $A^\dagger x_{(j)} = \lambda_j^* x_{(j)}$, so $x_{(j)}^\dagger A x_{(i)} = (\lambda_j^* x_{(j)})^\dagger x_{(i)} = \lambda_j (x_{(j)}^\dagger x_{(i)})$.

This says $(\lambda_j - \lambda_i) x_{(j)}^\dagger x_{(i)} = 0 \Rightarrow$ if $\lambda_j \neq \lambda_i$ (different eigenvalues), then $x_{(j)}$ and $x_{(i)}$ are orthogonal.

+ If one of the eigenvalues is degenerate, there are multiple eigenvectors with that eigenvalue. We can use the Gram-Schmidt procedure to make them orthonormal. That means we get N orthonormal eigenvectors, and that makes an orthonormal basis.

- Hermitian and Unitary Matrix eigenvalues (in complex vector space)
  + The eigenvalues of a Hermitian matrix are real. Consider eigenvector x_i with eigenvalue λ_i. Then $x_{(i)}^\dagger A x_{(i)} = \lambda_i x_{(i)}^\dagger x_{(i)}$ ($=\lambda_i$ if normalized) but also $x_{(i)}^\dagger A x_{(i)} = (A^\dagger x_{(i)})^\dagger x_{(i)} = (A x_{(i)})^\dagger x_{(i)} = \lambda_i^* x_{(i)}^\dagger x_{(i)} \Rightarrow \lambda_i^* = \lambda_i$
  + Similarly, anti-Hermitian matrices have pure imaginary eigenvalues.
  + The eigenvalues of a unitary matrix have $|\lambda|^2 = 1$. The reason is
  $$x_{(i)}^\dagger x_{(i)} = x_{(i)}^\dagger (A^\dagger A) x_{(i)} = (A x_{(i)})^\dagger (A x_{(i)}) = \lambda_i^* \lambda_i x_{(i)}^\dagger x_{(i)} \Rightarrow |\lambda_i|^2 = 1$$
- Simultaneous eigenbasis: when do 2 normal matrices have the same basis of eigenvectors?
  + First, suppose that A and B have common eigenvectors $x_{(i)}$ with respective eigenvalues λ_i and ρ_i. Then
  $$AB x_{(i)} = A(\rho_i x_{(i)}) = \rho_i (A x_{(i)}) = \rho_i \lambda_i x_{(i)} \quad \text{and}$$
  $$BA x_{(i)} = B(\lambda_i x_{(i)}) = \lambda_i (B x_{(i)}) = \lambda_i \rho_i x_{(i)} = AB x_{(i)}$$
  This is true on all the basis vectors $x_{(i)}$, so it is true on all vectors, so $AB = BA$. <u>If two normal matrices have the same eigenbasis, they commute</u>.
  + Suppose $AB = BA$. Then if $x_{(i)}$ is an eigenvector of A w/ eigenvalue λ_i, $AB x_{(i)} = B(A x_{(i)}) = \lambda_i B x_{(i)}$. Therefore $(B x_{(i)})$ is also an eigenvector with that eigenvalue. This must be proportional to $x_{(i)}$, so $B x_{(i)} = \rho_i x_{(i)}$, so $x_{(i)}$ is also an eigenvector of B. $A + B$ share eigenvectors. (The proof is slightly more complicated with degeneracy).