- Inverse Matrix and Systems of Equations

- Remember that linear operators $A: X \to X$ can have inverse operators A^{-1} s.t. $A^{-1}A = AA^{-1} = \mathbb{1}$. These operators correspond to square matrices, so square matrices can have inverses
  - + It will turn out that non-singular matrices (those with inverses) must have $\det \neq 0$. Singular matrices ($\det = 0$) have no inverse.
  - + If the inverse matrix of A exists, it is unique: Suppose $AB = BA = \mathbb{1}$ and $AC = CA = \mathbb{1}$. Start with $AB = \mathbb{1}$ and multiply both sides on the left with C. Then $CAB = (CA)B = \mathbb{1}B = B = C\mathbb{1} = C$, so $B = C = A^{-1}$.
  - + Can you prove the following basic properties of the inverse easily?
    (a) $(A^{-1})^{-1} = A$ (b) $(A^T)^{-1} = (A^{-1})^T$ (c) $(A^\dagger)^{-1} = (A^{-1})^\dagger$ (d) $(A_1 A_2 \cdots A_n)^{-1} = A_n^{-1} \cdots A_2^{-1} A_1^{-1}$
    (e) $\det(A^{-1}) = 1/\det(A)$
- The relation to systems of linear equations: We often encounter M linear equations of N unknowns

$$
\begin{aligned}
A_{11}x_1 + A_{12}x_2 + \cdots + A_{1N}x_N &= b_1 \\
A_{21}x_1 + A_{22}x_2 + \cdots + A_{2N}x_N &= b_2 \\
&\vdots \\
A_{M1}x_1 + A_{M2}x_2 + \cdots + A_{MN}x_N &= b_M
\end{aligned}
$$

We should recognize this as a matrix multiplication $Ax = b$ where A is an $M \times N$ matrix with elements given by the coefficients, x is an N-dim column with $x^T = [x_1\ x_2 \cdots x_N]$, and b is an M-dim column with $b^T = [b_1\ b_2 \cdots b_M]$
  - + For $b \neq 0$, $Ax = b$ is an inhomogeneous equation. There is always a corresponding homogeneous equation $Ax = 0$.
  - + When are there solutions? Call the columns of A individual vectors

$$|a_1\rangle \equiv a_1 = \begin{bmatrix} A_{11} \\ A_{21} \\ \vdots \end{bmatrix}, \quad \begin{matrix}|a_2\rangle \equiv \\ a_2 \equiv\end{matrix} \begin{bmatrix} A_{12} \\ A_{22} \\ \vdots \end{bmatrix}, \text{ etc.}$$

Then the system of equations can be written (in abstract form)

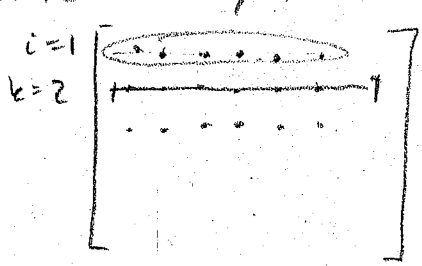
$$x_1|a_1\rangle + x_2|a_2\rangle + \cdots + x_N|a_N\rangle = |b\rangle$$

In other words, there is a solution iff $|b\rangle$ is in the span of $\{|a_1\rangle, |a_2\rangle, \ldots\}$.

+ Sometimes $Ax=b$ has a unique solution (only one). However, if there is a solution $y \neq 0$ to the homogeneous equation $Ay=0$ then $A(x+y) = Ax + 0 = b$ shows $(x+y)$ solves $Ax=b$ also. There can be an infinite number of solutions.

- Solution methods for N equations of N unknowns are also ways to invert a sq. matrix
  + If $\det(A) = 0$, A is singular, and $Ax=0$ has a nonzero solution ($Ax=b$ has infinite solutions)
  + However, if $\det(A) \neq 0$, matrix inversion says $x = A^{-1}b$ solves the system.
  + So figuring out how to invert the matrix and solve $Ax=b$ are the same question.

- General formula for the inverse.
  + Call the cofactors of element $A_{ij} = \mathrm{cof}(A)_{ij}$. These make matrix $\mathrm{cof}(A)$
    $A^{-1} = \mathrm{cof}(A)^T/\det(A)$. To see this, note that
    $A_{ij}\,\mathrm{cof}(A)_{kj}$ for $i=k$ is a cofactor expansion of $\det(A)$ along row i. If $i \neq k$, the sum is the determinant of a matrix with 2 identical rows, which $=0$.



+ For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, this is $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

+ Solving $Ax=b$ using this method to find A^{-1} is the same as using <u>Cramer's rule</u>.

- Gaussian Elimination (briefly)

  + We can solve a system of linear equations by repeatedly (a) multiplying equations by a constant (b) adding the multiple of one equation to another and (c) reorganizing equations to put them in the form

$$x_1 + 0 + \cdots = c_1$$
$$0 + x_2 + 0 \cdots = c_2$$

  + This is the same as performing those row operations on the augmented matrix

$$\left[\begin{array}{cccc|c} A_{11} & A_{12} & \cdots & A_{1N} & b_1 \\ A_{21} & & \cdots & & b_2 \\ \vdots & & \ddots & & \vdots \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & \cdots & & c_1 \\ 0 & 1 & 0 & \cdots & c_2 \\ 0 & 0 & 1 & \cdots & \vdots \\ & & & \ddots & \vdots \end{array}\right]$$

  + Suppose $b = [1\ 0 \cdots]^T$. Then the vector c is the column that gives $\begin{bmatrix} 1 \\ 0 \\ \vdots \end{bmatrix}$ when multiplied by A. Similarly for $b = [0\ 1\ 0 \cdots]^T$.

    So then the Gaussian elimination

$$\left[A \mid \mathbb{1} \right] \longrightarrow \left[\mathbb{1} \mid A^{-1} \right]$$ gives the inverse.

- LU decomposition

  + It turns out that it is always possible to write $A = LU$, where L is <u>lower triangular</u> (and 1 on the diagonal) and U is <u>upper triangular</u>

$$L = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ L_{21} & 1 & 0 & \cdots \\ L_{31} & L_{32} & 1 & 0 \cdots \\ \vdots & & & \ddots \end{bmatrix}, \quad U = \begin{bmatrix} U_{11} & U_{12} & \cdots \\ 0 & U_{22} & \cdots \\ 0 & 0 & U_{33} \cdots \\ & & \ddots \end{bmatrix}$$

+ You can solve for the elements U_{ij} and L_{ij} by doing the matrix multiplication and solving 1 element at a time. In the 3x3 case,

$$\begin{bmatrix} A_{11} & A_{12} & \cdots \\ & \ddots & \\ & & \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12}+U_{22} & L_{21}U_{13}+U_{23} \\ L_{31}U_{11} & L_{31}U_{12}+L_{32}U_{22} & L_{31}U_{13}+L_{32}U_{23}+U_{33} \end{bmatrix}$$

so $U_{11}=A_{11}$, $L_{21}=A_{21}/A_{11}$, etc.

+ Then solve $Ly=b$ as you like (Gaussian elimination works well b/c of the triangular form) and then $Ux=y$ to get the solution x.

+ This also allows you to find $A^{-1}=U^{-1}L^{-1}$ fairly easily.