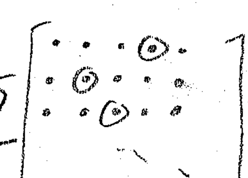
– Functions of Square Matrices

- Power Series
  + A square matrix can multiply itself (ie, $(N\times N)(N\times N) = (N\times N)$), so we can raise it to a power $A^n = \underbrace{A A \cdots A}_{n \text{ times}}$
  + Many functions can be written as a Taylor series, ie, $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n f}{dx^n}(0)\, x^n = \sum f_n x^n$. Give some examples.
  + We can define a function $f(A)$ for a square matrix A in terms of the Taylor series of $f(x)$.
    $$f(A) = \sum_n^{\infty} f_n A^n$$
- Trace: a single number
  + The elements $A_{11}, A_{22}, \ldots, A_{NN}$ lie on the main diagonal of a square matrix
    $$\begin{bmatrix} A_{11} & A_{12} & & \\ A_{21} & A_{22} & & \\ & & \ddots & \\ & & & A_{NN} \end{bmatrix}$$
  + The trace is the sum of the diagonal elements $tr(A) = A_{ii}$.
  + The trace is a linear operation: $tr(A+B) = tr A + tr B$ and $tr(\lambda A) = \lambda\, tr A$.
  + Cyclic Property: The trace is unchanged by cyclic permutations $tr(A_1 A_2 \cdots A_n) = tr(A_n A_1 \cdots A_{n-1})$. To understand this, note that
    $$tr(AB) = (AB)_{ii} = A_{ij} B_{ji} = B_{ji} A_{ij} = (BA)_{jj} = tr(BA)$$
  + Can you show $tr(A^T) = tr(A)$ and $tr(A^\dagger) = tr(A)^*$?
- Determinant: another single number
  + The determinant is given by a sum of products of one term per row, with a sign. It is defined

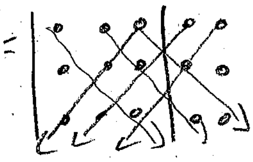
$$\det(A) \equiv |A| \equiv \sum_P (-1)^{sign(P)} A_{1j_1} A_{2j_2} \cdots A_{Nj_N} \approx \sum$$



where P is a permutation $\{j_1, j_2, \ldots j_N\}$ of $\{1, 2, 3, \ldots N\}$. $Sign(P) = +1$ if $\{j_1, j_2, \ldots j_N\} = \{1, 2, \ldots N\}$ and changes sign each time a value is permuted.

+ Smaller matrices are fairly simple:

2x2: $\det A = A_{11}A_{22} - A_{12}A_{21}$. Graphically, $\det =$  is a good mnemonic

3x3: $\det A = A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32}$
$- A_{11}A_{23}A_{32} - A_{12}A_{21}A_{33} - A_{13}A_{22}A_{31}$ $\det =$ 

You can see that larger one will rapidly get complicated.

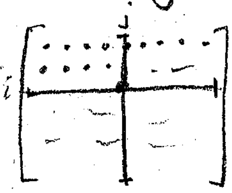
+ The way to calculate larger ones is to notice that we can write (reorganizing the sum)

$$\det(A) = A_{11}C_{11} + A_{12}C_{12} + \cdots + A_{1N}C_{1N}$$

where each of the factors C_{ij} looks like a smaller determinant

$$C_{11} = \sum(\pm) A_{2j_2}A_{3j_3} \cdots A_{Nj_N} \quad \text{for example}$$

+ In fact, the <u>cofactor</u> C_{ij} of the ij element is $(-1)^{i+j}$ times the determinant of the $(N-1)\times(N-1)$ matrix formed by removing the i^{th} row and j^{th} column of **A**:

$C_{ij} = (-1)^{i+j} \det$ 

+ Then $\det(A)$ is given by summing the element × the cofactor either across <u>any</u> row or down <u>any</u> column.

$$\det(A) = A_{11}C_{11} + A_{12}C_{12} + \cdots + A_{1N}C_{1N}$$
$$= A_{21}C_{21} + A_{22}C_{22} + \cdots + A_{2N}C_{2N} = \cdots$$
$$= A_{11}C_{11} + A_{21}C_{21} + \cdots + A_{N1}C_{N1} = \cdots$$

This is still difficult usually, and computers help a lot.

+ There are a few simple properties of the determinant:

$\det(A^T) = \det(A)$, $\det(A^*) = \det(A^\dagger) = \det(A)^*$, $\det(\lambda A) = \lambda^N \det(A)$,

the determinant changes sign if you interchange 2 rows or 2 columns

+ These properties (unproved here) can help evaluating determinants:

  a) if 2 rows (or 2 columns) are multiples of each other, $\det(A) = 0$.

  b) if you add a multiple of one row (or column) to another, the determinant is unchanged

  c) $\det(AB) = \det(A)\det(B)$

+ One strategy for a large determinant is to add multiples of other rows (or columns) to a particular row (or column) in order to reduce as many elements as possible to zero. That simplifies the sum over cofactors

+ Work some examples.