Matrix Algebra

- What is a matrix? A set of numbers $B_{i,j}$ with $i=1,2,\ldots M$ and $j=1,\ldots N$ arranged in a 2D array is the matrix B. (Normally, we drop the comma in the subscript)

$$B = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1N} \\ B_{21} & B_{22} & \cdots & B_{2N} \\ \vdots & \vdots & & \vdots \\ B_{M1} & B_{M2} & \cdots & B_{MN} \end{bmatrix}.$$

Elements may be complex.

- Since column vectors form a vector space, we can represent <u>any</u> N-dim. vector space with N-dim columns
  + Choose a basis $\{|e_1\rangle, |e_2\rangle, \ldots |e_N\rangle\}$. Map these to column vectors

$$|e_1\rangle \to \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad |e_2\rangle \to \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad |e_i\rangle \to i^{th} \to \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots \quad |e_N\rangle \to \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

  + Then any vector $|x\rangle$ with components x_i in this basis is

$$|x\rangle = x_i |e_i\rangle \to \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

  + <u>Be careful</u>: just like the components change if we change the basis, so does the column representation!

- Matrix Multiplication

  It is also possible to multiply matrices together
  - Start with $M \times N$ matrix A and $N \times P$ matrix B. Then $C = AB$ is an $M \times P$ matrix
    + We define the elements of C by "summing a row with a column"

      Pictorially:

$$\begin{array}{cc} & j \\ i & \begin{bmatrix} C_{11} & \cdots & \\ \vdots & \boxed{C_{ij}} & \\ & & \end{bmatrix} \end{array} = \begin{array}{cc} & \\ i & \begin{bmatrix} A_{11} & & \\ \vdots & & \\ \boxed{A_{i1} \; A_{i2} \cdots A_{iN}} & & \\ & & \end{bmatrix} \end{array} \begin{array}{c} j \\ \begin{bmatrix} B_{11} & \cdots & \boxed{\begin{matrix} B_{1j} \\ B_{2j} \\ \vdots \\ B_{Nj} \end{matrix}} \\ \vdots & & \end{bmatrix} \end{array}$$

$$= \begin{array}{cc} & j \\ i & \begin{bmatrix} & & \\ & \boxed{A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots} & \\ & & \end{bmatrix} \end{array}$$

  + We can more efficiently write this in index notation for the matrix elements

$$C_{ij} = \sum_{k=1}^{N} A_{ik} B_{kj} = A_{ik} B_{kj}$$

- Some properties (assuming all multiplications below are defined)
  - Distributive: $C(A+B) = CA + CB$ and $(A+B)C = AC + BC$
  - Associative: $A(BC) = (AB)C$
  - But it is <u>not</u> commutative. For A an $M\times N$ matrix and B an $N\times P$ matrix, AB exists but BA does not. But even for A an $M\times N$ matrix and B an $N\times M$ matrix, AB and BA both exist but are different sizes! And even if A and B are both $N\times N$ matrices, AB and BA are the same size but <u>not necessarily equal</u>.
  - The $N\times N$ <u>identity matrix</u> is $\mathbb{1}_N = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix} \equiv \mathbb{1}$

    (ie, zero everywhere except 1 on the diagonal)

    and satisfies $\mathbb{1}_M A = A$ and $A\mathbb{1}_N = A$ for A an $M\times N$ matrix
  - For an $N\times 1$ column x, $Ax = y$, an $M\times 1$ column. A special example is a basis vector

$$\begin{bmatrix} A_{11} & - & A_{1i} & - \\ \vdots & & A_{2i} & \\ | & & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}_i = \begin{bmatrix} A_{1i} \\ A_{2i} \\ \vdots \\ A_{Mi} \end{bmatrix}$$

The product is the i^{th} column of A.

  - Similarly, a $1\times M$ row multiplied by an $M\times N$ matrix gives a $1\times N$ row. Multiplying by the i^{th} basis row $[0 \cdots 0\, 1\, 0 \cdots]$ yields the i^{th} row of the $M\times N$ matrix.
  - Get some practice!
  - The transpose (or adjoint) of a product is the product of the transposes (or adjoints): say $C = AB$.

    Then $(C^\dagger)_{ij} = C^*_{ji} = A^*_{jk} B^*_{ki} = (B^\dagger)_{ik}(A^\dagger)_{kj} \Rightarrow C^\dagger = B^\dagger A^\dagger$

    (and similarly for the transpose). In general

$$(A_1 A_2 \cdots A_M)^T = A_M^T \cdots A_2^T A_1^T \quad \text{and} \quad (A_1 A_2 \cdots A_M)^\dagger = A_M^\dagger \cdots A_2^\dagger A_1^\dagger$$

- Matrix multiplication can be used for inner products + linear operators:
  + Remember we can convert vectors to columns $|x\rangle \to x$ in a particular basis. The entries of the column are the components.
  + For an orthonormal basis, $\langle a|b\rangle = a_i^* b_i$. But this is also the matrix multiplication $a^\dagger b$. A row x a column = a number.
  + In a general basis, remember $\langle a|b\rangle = a_i^* G_{ij} b_j$. Note $(Gb)_i = G_{ij} b_j$, so $\langle a|b\rangle = a^\dagger G b$.
  + Let's check: $\langle b|a\rangle = b^\dagger G a = (a^\dagger G^\dagger b)^* = \langle a|b\rangle^*$ (adjoint of scalar = conjugate) if $G^\dagger = G$. That is just $G_{ji}^* = G_{ij}$
  + Let's remember also that the action of a linear operator $|y\rangle = A|x\rangle$ is $y_i = A_{ij} x_j$ in components. Just by comparison, this is matrix multiplication $y = Ax$ where A is the matrix with elements A_{ij}.
  + Summary: In a fixed basis, an inner product is row x column multiplication, and action with a linear operator is multiplication by a matrix with the same elements