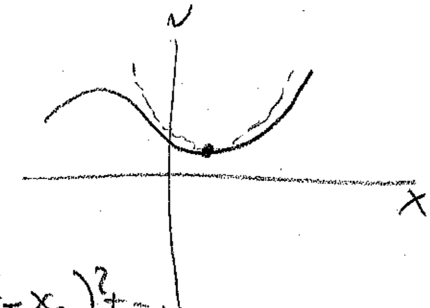
The Harmonic Oscillator

$V = \frac{1}{2} m\omega^2 x^2$ Classical motion: $x(t) = Ae^{i\omega t} + Be^{-i\omega t}$

- Why is this so important?
  - Look near a minimum of any potential
  - Taylor expand



$$V(x) \simeq V(x_0) + V'(x_0)(x-x_0) + \frac{1}{2}V''(x_0)(x-x_0)^2 + \dots$$

  - The constant really doesn't matter (except in gravity)
  - The 1st derivative vanishes because this is a minimum

So <u>any</u> potential looks like a harmonic oscillator near its minimum $V(x) \simeq \frac{1}{2}V''(x_0)(x-x_0)^2$ (if $V''(x_0) \neq 0$)

This is how we do most of particle physics, for example.

- Solution by differential equations.
  - The Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 x^2 \psi = E\psi$$

  - Rewrite $\xi = \sqrt{\frac{m\omega}{\hbar}}\, x$, $K = 2E/\hbar\omega$

$$\frac{d^2\psi}{d\xi^2} = (\xi^2 - K)\psi$$

  - At $\xi^2 \gg K$, $d^2\psi/d\xi^2 \simeq \xi^2\psi$, so $\psi \simeq Ae^{-\xi^2/2} + Be^{\xi^2/2}$
    - Work out the second derivative to check this

+ Clearly, the 2^{nd} choice is non-normalizable, so we don't want that

- To make this explicit, write $\psi(\xi) = h(\xi)\, e^{-\xi^2/2}$
  + Then
$$\frac{d^2h}{d\xi^2} - 2\xi\frac{dh}{d\xi} + (K-1)h = 0$$
(see text for algebra)
  + To solve for h, write it as a series
$$h(\xi) = \sum_{j=0}^{\infty} a_j \xi^j = a_0 + a_1\xi + a_2\xi^2 + \cdots$$
No negative powers allowed if we want normalizability
  + The differential equation becomes
$$\sum_j \left[(j+1)(j+2)a_{j+2} - 2ja_j + (K-1)a_j \right] \xi^j = 0$$
- The solution is any series that obeys the <u>recursion relation</u>
$$a_{j+2} = \frac{2j+1-K}{(j+1)(j+2)}a_j$$
  + This determines even coefficients in terms of a_0, odd in terms of a_1 ⟵ two solutions, as expected.
  + At large j, $a_{j+2} \simeq 2a_j/j$. Notice $e^{\xi^2} = \sum_n \frac{1}{n!}\xi^{2n} = \sum_j \frac{1}{(j/2)!}\xi^j$ has the same relationship at large j

    Then the wavefunction $\psi = he^{-\xi^2/2} \simeq e^{\xi^2/2}$ ⟵ not normalizable
  + To get the normalizable solution
    1) Either the even or odd series must be zero
    2) The other series must terminate after some j.

    That requires $K = 2j+1$ for some j

    $\Longrightarrow E = \hbar\omega(n + \tfrac{1}{2})$ for some positive integer n

+ The general normalizable solution is

$$\psi_n = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2}$$

where H_n is the Hermite Polynomial. See table 2.1

- Solution by Operators
  - Define the operator $a = \sqrt{\frac{m\omega}{2\hbar}}\, x + i \frac{p}{\sqrt{2\hbar m\omega}}$ and its adjoint $a^\dagger$
    + The commutator

$$[a, a^\dagger] = \left(\frac{1}{2\hbar}\right)(-2i)[x,p] = 1$$

    + The inverse relation is $x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)$, $p = i\sqrt{\frac{\hbar m\omega}{2}}(a^\dagger - a)$
  - Then the Hamiltonian is

$$H = \frac{1}{2m}\left[p^2 + (m\omega x)^2\right] = \frac{\hbar\omega}{4}\left[-(a^{\dagger 2} + a^2 - a^\dagger a - a a^\dagger) + (a^{\dagger 2} + a^2 + a a^\dagger + a^\dagger a)\right]$$

$$= \frac{\hbar\omega}{2}(a a^\dagger + a^\dagger a) = \hbar\omega\left(a^\dagger a + \tfrac{1}{2}\right)$$

    + We get the commutators

$$[a, H] = \hbar\omega a, \quad [a^\dagger, H] = -\hbar\omega a^\dagger$$

    + This means $a|E\rangle$ and $a^\dagger|E\rangle$ are eigenstates if $|E\rangle$ is

$$H(a|E\rangle) = a(H - \hbar\omega)|E\rangle = (E - \hbar\omega)a|E\rangle \leftarrow \text{lowers energy}$$

$$H(a^\dagger|E\rangle) = (E + \hbar\omega)a^\dagger|E\rangle \leftarrow \text{raises energy}$$

    + a and $a^\dagger$ are ladder operators called lowering (or annihilation) operator and raising (or creation) operator
  - You can't lower the energy forever: remember that normalizable states have $E > V_{min} = 0$.

+ There must be some state such that $a|0\rangle = 0$.

This state has $H|0\rangle = \hbar\omega(a^\dagger a + \frac{1}{2})|0\rangle = \frac{\hbar\omega}{2}|0\rangle$, or $E = \frac{\hbar\omega}{2}$

It is the lowest energy state = ground state.

Note: the energy, as in other systems, is not zero.

+ The other states come from repeated application of $a^\dagger$.

The state $|n\rangle$ with energy $E_n = \hbar\omega(n+\frac{1}{2})$ has clearly $a^\dagger a|n\rangle = n|n\rangle$

By the commutator $aa^\dagger|n\rangle = (n+1)|n\rangle$.

But also $a^\dagger|n\rangle = c_n|n+1\rangle$.

Normalizing $\langle n|aa^\dagger|n\rangle = c_n^2 = (n+1) \Rightarrow a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$

Similarly $a|n\rangle = \sqrt{n}\,|n-1\rangle$

+ Eigenstates are therefore

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle$$

That solves it completely.