Square Wells

- Let's start with the infinite square well $V(x) = \begin{cases} 0 & -a<x<a \\ \infty & |x|>a \end{cases}$

  (Using slightly different notation than Griffiths)

  - The potential just imposes Dirichlet b.c. at $x = \pm a$. (ψ cannot penetrate ∞ potential), Otherwise, the Hamiltonian is just a free particle for $-a<x<a$.
  - The general solution is $\psi = A\cos kx + B\sin kx$, $k = \sqrt{2mE}/\hbar$.

    + To satisfy the Dirichlet bc, we can have

    $$k = 0, \pm\pi/2a, \pm\pi/a, \pm 3\pi/2a, \pm 2\pi/a, \ldots$$

    alternating sine + cosine solutions.

    + $k=0$ is not a solution because $\sin(0 \cdot x) = 0$.

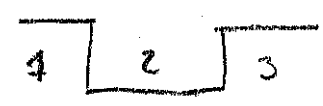
    Also, k negative are not independent b/c cos. + sin are even/odd

    + So the wave functions are

    $$\left.\begin{aligned} \psi &= \frac{1}{\sqrt{a}}\cos\left(\frac{n\pi}{2a}x\right), \ n \text{ odd} \\ \psi &= \frac{1}{\sqrt{a}}\sin\left(\frac{n\pi}{2a}x\right), \ n \text{ even} \end{aligned}\right\} \quad E_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{2a}\right)^2$$

  - The solutions satisfy all the nice properties of orthonormality, etc. through the usual ideas about Fourier series.

- The finite square well $V(x) = \begin{cases} 0 & x<-a \\ -V_0 & -a<x<a \\ 0 & a<x \end{cases}$



  - Solutions will be either even or odd for bound states. We will look only at the even ones.

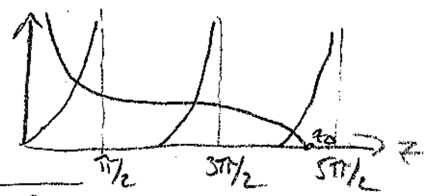
    + If P is the operator $P|x\rangle = |-x\rangle$ (ie, $P\psi(x) = \psi(-x)$) we see $[H,P]=0$. So there is a simultaneous eigenbasis. Convince yourself of this statement! (Hint: $Px = -xP$)

+ The <u>ground state</u> (lowest energy) has no nodes - places where probability density or wavefunction $= 0$. In general, the more nodes, the higher the energy

- The even states are

$$\psi = \begin{cases} A e^{\kappa x} & x < -a \\ B\cos(k' x) & -a < x < a \\ A e^{-\kappa x} & x > a \end{cases} \qquad \kappa = \sqrt{-2mE}/\hbar \qquad k' = \sqrt{2m(E+V_0)}/\hbar$$

  + The bound state energy $E > V_0$ because energy > minimum potential
  + Continuity of ψ requires $Ae^{-\kappa a} = B\cos(k'a)$
    Continuity of $d\psi/dx$ requires $-\kappa A e^{-\kappa a} = -k' B \sin(k'a)$
  + Therefore, $\kappa = k' \tan(k'a)$
    As in text, note $\kappa^2 + (k')^2 = 2mV_0/\hbar^2$.
    Define $z = k'a$, $z_0 = \frac{a}{\hbar}\sqrt{2mV_0}$. Then $\tan z = \sqrt{(z_0/z)^2 - 1}$



  + there is always one even bound state, but the total number depends on z_0

- Scattering states
  + We now need

$$\psi = \begin{cases} Ae^{ikx} + Be^{-ikx} & \text{region 1} \\ C\sin(k'x) + D\cos(k'x) & \text{region 2} \\ E e^{ikx} & \text{region 3} \end{cases} \qquad k = \sqrt{2mE}/\hbar \qquad k' = \sqrt{2m(E+V_0)}/\hbar$$

  + Must also consider b.c. at both $x = \pm a$ separately: b/c this is a scattering state, we can't make ψ odd or even
  + After some algebra,

$$T^{-1} = 1 + \frac{V_0^2}{4E(E+V_0)} \sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E+V_0)}\right)$$

This is transparent $T = 1$ when the sine $= 0$ or

$$E + V_0 = \frac{\hbar^2}{2m}\left(\frac{n\pi}{2a}\right)^2 \leftarrow \text{energy of infinite square well bound state above minimum}$$

The phenomenon of increased transmission related to specific energies is resonance. (This is "above barrier" resonance.)

- A few words on the probability current and reflection/transmission coeffs. We should think of $\vec{J}$ as probability flowing through a (transverse) unit area per unit time. This is very much like a charge current. For a truly stationary (ie bound) state, $\vec{J}$ must be divergenceless. But in scattering states it is telling us about the probability to detect a particle sent back to $x \to -\infty$ or transmitted to $x \to +\infty$