- Probabalistic Interpretation (3rd axiom)

In classical mechanics, the value of every observable is determined by the state.

☆ In QM, if the system is in state $|\psi\rangle$, then the probability of measuring the eigenvalue λ of observable/operator $\mathcal{O}$ with eigenstate $|\lambda\rangle$ is $P_\lambda = |\langle\lambda|\psi\rangle|^2$

- The meaning of the probability:
  - We often think of probability as the fraction of times some result comes up in repeated measurements
    - In QM, this is <u>not</u> repeated measurements of the same system
    - The probability is based on measurements made on many systems prepared identically.
  - At the time of a measurement, we say the "wave function collapses" $|\psi\rangle \to |\lambda\rangle$ where $|\lambda\rangle$ is the eigenstate associated w/ the measured value. We'll discuss the "controversy" about this later this year

- Consequences:
  - Probabilities do, in fact, add to unity
    - By the completeness relation (for discrete eigenvalues)

$$\sum_\lambda P_\lambda = \sum_\lambda \langle\psi|\lambda\rangle\langle\lambda|\psi\rangle = \langle\psi|\psi\rangle = 1$$

    - This means wavefunctions represent <u>probability densities</u>

$$\int d^3\vec{x}\, P(\vec{x}) = \int d^3\vec{x}\, |\langle\vec{x}|\psi\rangle|^2 = \int d^3\vec{x}\, |\psi(\vec{x})|^2 = 1$$

    - Therefore, the probability of finding the particle in volume V is

$$P_V = \int_V d^3\vec{x}\, |\psi(\vec{x})|^2$$

Note: You can't find the particle exactly at one point anyway. You always have to deal with detector resolution.

+ This is similar for other continuous observables (like momentum in some cases)

- We can also look at average values over our repeated experiments.
  + The <u>expectation value</u> (or mean) of a given observable is

$$\langle \mathcal{O} \rangle = \sum_\lambda \lambda P_\lambda = \langle \psi | \left(\sum_\lambda \lambda |\lambda\rangle\langle\lambda| \right) |\psi\rangle = \langle \psi | \mathcal{O} | \psi \rangle$$

So a "diagonal matrix element" of $\mathcal{O}$ gives its expectation value

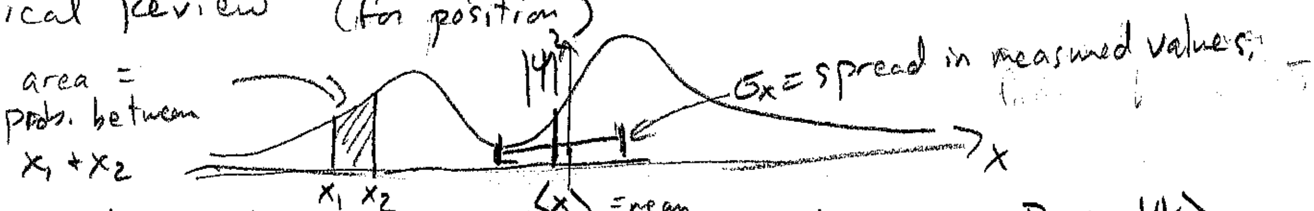
  + Measurements fall in a probability distribution, so we could ask about higher moments. The most important is the <u>uncertainty</u> (standard deviation) which we get by taking

$$\sigma_{\mathcal{O}}^2 = \langle \Delta \mathcal{O}^2 \rangle \quad \text{with} \quad \Delta \mathcal{O} \equiv \mathcal{O} - \langle \mathcal{O} \rangle$$

Note: It is often easiest to calculate $\sigma_{\mathcal{O}}$ by using

$$\langle \Delta \mathcal{O}^2 \rangle = \langle \mathcal{O}^2 \rangle - 2\langle \mathcal{O} \langle \mathcal{O} \rangle \rangle + \langle \mathcal{O} \rangle^2 = \langle \mathcal{O}^2 \rangle - \langle \mathcal{O} \rangle^2$$

- Graphical Review (for position)



- Note: a phase makes no difference in a state. $e^{i\phi}|\psi\rangle$ has same P_λ as $|\psi\rangle$.

Heisenberg Uncertainty Principle: a famous + important relationship often written as $\Delta x \, \Delta p \geq \hbar/2$

- The meaning relates to $\vec{p} \simeq -i\hbar\vec{\nabla}$ as an operator.
  + For a p eigenstate, $p = 2\pi\hbar/\lambda$, so an error Δp relates to an error $\Delta\lambda$. (or spread $\Delta\lambda$) (λ = wave length)
  + But a well-localized wave has a large wavelength spread, so there is a trade off

well-defined λ vs well-defined position.

+ Note: It is <u>not</u> really about the momentum of light in a microscope kicking around the particle you observe — that's successive measurements. Remember ΔO is the spread of measurements in identically-prepared systems.

- Precisely, take $\Delta x = \sigma_x$, $\Delta p = \sigma_p$, std. dev. of measurements on that ensemble.

  + For any observables A + B, in a state $|\psi\rangle$, we can write

$$\sigma_A^2 = \langle\psi|(\Delta A)^2|\psi\rangle = \langle f|f\rangle \,,\ |f\rangle = \Delta A|\psi\rangle$$
$$\sigma_B^2 = \langle g|g\rangle \,,\ |g\rangle = \Delta B|\psi\rangle .$$

  + Then the <u>Schwarz inequality</u> says

$$\sigma_A^2 \sigma_B^2 = \langle f|f\rangle\langle g|g\rangle \geq |\langle f|g\rangle|^2$$

  (Basically, this is $|\langle f|g\rangle| = \sqrt{\langle f|f\rangle}\sqrt{\langle g|g\rangle}\cos\theta$)

  + But $|\langle f|g\rangle|^2 \geq (\mathrm{Im}\langle f|g\rangle)^2 = -\frac{1}{4}(\langle f|g\rangle - \langle g|f\rangle)^2$

$$\Rightarrow \quad \sigma_A^2\sigma_B^2 \geq -\frac{1}{4}\left[\langle\psi|\Delta A\,\Delta B|\psi\rangle - \langle\psi|\Delta B\,\Delta A|\psi\rangle\right]^2$$

- Operators do not commute! Generally $AB|\psi\rangle \neq BA|\psi\rangle$

  + As an example consider x and p in position basis

$$\langle x|xp|\psi\rangle = x\left(-i\hbar\frac{d\psi}{dx}\right) \text{ vs } \langle x|px|\psi\rangle = -i\hbar\frac{d}{dx}(x\psi) = -i\hbar\psi + x\left(-i\hbar\frac{d\psi}{dx}\right)$$

  So $(xp-px)|\psi\rangle = +i\hbar|\psi\rangle$ <u>independent</u> <u>of state</u> $|\psi\rangle$

  That means you can define an operator <u>commutator</u> $[x,p] \equiv xp - px = +i\hbar$

  + Generally, $[A,B] \equiv AB - BA$ is the commutator. Note $[A,B] = -[B,A]$

- We get

$$\sigma_A^2\sigma_B^2 \geq -\frac{1}{4}\langle\psi|[A,B]|\psi\rangle^2 \quad \text{Comments:}$$

  + $[\Delta A, \Delta B] = [A,B]$ (You can show it yourself)

  + $[A,B]^\dagger = -[A,B]$ for Hermitian operators, so its eigenvalues + expectation values are pure imaginary

+ The Heisenberg uncertainty principle is $\sigma_x \sigma_p \geq \hbar/2$

+ We have derived a general uncertainty principle valid for any operators.

- Wave functions of minimal x-p uncertainty $\sigma_x \sigma_p = \hbar/2$ are Gaussian $\psi(x) \propto e^{-ax^2}$. More later