Addition of Angular Momentum

What if you have multiple particles? Or one with $\vec{L}$ and $\vec{S}$?
What is the total angular momentum?

- An example: 2 spin 1/2 particles, like 2 electrons
  + First, we have to split states into factors for the 2 spins

$$|+\rangle|+\rangle, \quad |+\rangle|-\rangle, \quad |-\rangle|+\rangle, \quad |-\rangle|-\rangle \quad \text{or linear combinations}$$

  + Total spin operator is $\vec{S} = \vec{S}^{(1)} + \vec{S}^{(2)}$; (1) or (2) means that it acts on the 1st or 2nd factor.
  + The total z-component just adds

$$S_z|\psi\rangle = (S_z^{(1)}|\psi_1\rangle)|\psi_2\rangle + |\psi_1\rangle(S_z^{(2)}|\psi_2\rangle)$$

For S_z eigenstates, $m = m_1 + m_2$

  + In our example, $S_z(|+\rangle|+\rangle) = (\frac{\hbar}{2}|+\rangle)|+\rangle + |+\rangle(\frac{\hbar}{2}|+\rangle) = \hbar|+\rangle|+\rangle$

Then (etc)

	+ +	+ −	− +	− −
m	+1	0	0	−1

  + This looks like a triplet ($m=-1,0,+1$) and a singlet ($m=0$)

But what is what? Consider

$$S_-|1,1\rangle = \sqrt{2}\hbar|1,0\rangle \quad \text{from homework.}$$

That is

$$S_-(|+\rangle|+\rangle) = (S_-^{(1)}|+\rangle)|+\rangle + |+\rangle(S_-^{(2)}|+\rangle) = \hbar(|-\rangle|+\rangle + |+\rangle|-\rangle)$$

You'd get the same thing by acting S_+ on $|-\rangle|-\rangle$

  + Looks like $s=1$ and $s=0$ states:

$$S_{tot}=1 \text{ triplet} \begin{cases} |1,1\rangle = |+\rangle|+\rangle \\ |1,0\rangle = \frac{1}{\sqrt{2}}(|+\rangle|-\rangle + |-\rangle|+\rangle) \\ |1,-1\rangle = |-\rangle|-\rangle \end{cases}$$

$$S_{tot}=0 \text{ singlet} \quad |0,0\rangle = \frac{1}{\sqrt{2}}(|+\rangle|-\rangle - |-\rangle|+\rangle)$$

orthogonal combination

+ Check total spin using $S^2 = (\vec{S}^{(1)} + \vec{S}^{(2)})^2 = S^{(1)2} + S^{(2)2} + 2\vec{S}^{(1)}\cdot\vec{S}^{(2)}$

This is relatively easy to work out w/ Pauli matrix form

$$\vec{S}^{(1)}\cdot\vec{S}^{(2)}(|+\rangle|-\rangle) = S_x^{(1)}|+\rangle S_x^{(2)}|-\rangle + S_y^{(1)}|+\rangle S_y^{(2)}|-\rangle + S_z^{(1)}|+\rangle S_z^{(2)}|-\rangle$$

etc.

You can follow the calculation in the text (remember $S^{(1)2} = \hbar^2 \frac{1}{2}\left(\frac{3}{2}\right) = \frac{3}{4}\hbar^2$ on any state, same for $S^{(2)2}$)

$$S^2|1,0\rangle = 2\hbar^2|1,0\rangle \text{ and } S^2|0,0\rangle = 0$$

- Generally add 2 angular momenta j_1 and j_2 (can be spins or orbital)
  - + Start with state $|j_1, m_1 = j_1\rangle|j_2, m_2 = j_2\rangle$, and lower with $\vec{J}_- = J_-^{(1)} + J_-^{(2)}$

    That gives you states $|j = j_1+j_2, m = j_1+j_2\rangle, |j_1+j_2, j_1+j_2-1\rangle, \ldots |j_1+j_2, -j_1-j_2\rangle$
  - + Then you find an orthogonal state with $m = j_1+j_2-1$ and repeat for $j = j_1+j_2-1$, etc

    2 states w/ this m
    linearly indep.
  - + Possible total angular momenta:

    $$j = j_1+j_2,\ j_1+j_2-1,\ \ldots\ |j_1-j_2| \quad \text{in steps of 1. One set of each.}$$
  - + We will omit the proof for a later class, but is it reasonable? Count # of states:

Product

j_1 has $2j_1+1$ states

j_2 has $2j_2+1$

Total is

$$(2j_1+1)(2j_2+1)$$

$$= 4j_1j_2 + 2j_1 + 2j_2 + 1$$

Sum

Each j has $2j+1$ states (assume $j_1 \geq j_2$)

Total is

$$\sum_{j=j_1-j_2}^{j_1+j_2} (2j+1) = 2\left[\frac{2j_1(2j_2+1)}{2}\right] + \left[2j_2+1\right]$$

$$= 4j_1j_2 + 2j_1 + 2j_2 + 1$$

+ Relates to the idea that angular momentum generates rotations (see HW), rotations form a group, and we are combining representations of that group. Understanding symmetries like rotations as groups is a key idea of 20th century physics

- Clebsch-Gordan Coefficients: Tabulated answers

+ Generally, combined angular momenta satisfy

$$|j,m\rangle = \sum_{\substack{m_1,m_2 \\ m_1+m_2=m}} C^{j_1 j_2 j}_{m_1 m_2 m} |j_1,m_1\rangle |j_2,m_2\rangle$$

where the constants $C^{j_1 j_2 j}_{m_1 m_2 m}$ are the Clebsch-Gordan coefficients.

These are tabulated in a standard but odd-looking way on pg 188 of Griffiths.

+ You can also invert the linear transformation + you get the same coefficients.

$$|j_1,m_1\rangle |j_2,m_2\rangle = \sum_{j=|j_1-j_2|}^{j_1+j_2} C^{j_1 j_2 j}_{m_1 m_2 m} |j, m=m_1+m_2\rangle$$

+ These relate the basis of states of total angular momentum $|j,m;j_1,j_2\rangle$ to the factorized basis $|j_1,m\rangle|j_2,m\rangle$

Note that $\vec{J}^{(1)2}$ and $J^{(2)2}$ commute with $\vec{J}$.