

PHYS-4601 Homework 7 Due 29 Oct 2015

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. Odd States of the Finite Square Well from Griffiths 2.29

Consider the finite square well potential

$$V(x) = \begin{cases} 0 & \text{for } x < -a \\ -V_0 & \text{for } -a < x < a \\ 0 & \text{for } a < x \end{cases}, \quad (1)$$

as in class. Find the transcendental equation that determines the energy eigenvalues for bound states with *odd* wavefunctions. For a given V_0 and a , how many odd bound states are there?

2. Numerical Determination of Energy Eigenvalue related to Griffiths 2.51

Consider the potential

$$V(x) = -\frac{\hbar^2 a^2}{m} \frac{1}{\sqrt{1+a^2 x^2}}. \quad (2)$$

Find the ground state energy using the numerical “wag the dog” method with Maple, as follows:

- (a) Show that the Schrödinger equation for a bound state can be written in terms of dimensionless variables as

$$\frac{d^2 \psi}{d\xi^2}(\xi) + \left(\frac{2}{\sqrt{1+\xi^2}} - \kappa^2 \right) \psi(\xi) = 0, \quad (3)$$

where κ is a positive constant.

- (b) The ground state should be an even function, so if we ignore normalization, we can choose initial conditions $\psi(0) = 1$, $\psi'(0) = 0$. Starting with $\kappa^2 = 2.0$, enter the Schrödinger equation and initial conditions into Maple and solve the ODE numerically over the range $\xi = 0 - 10$. Then plot the solution. You may find the following Maple code helpful (note that we rename variables for ease of typing):

```
with(plots):
  schr := diff(u(x),x$2)+(2/sqrt(1+x^2)-2.00)*u(x) = 0
  init1 := u(0) = 1
  init2 := (D(u))(0) = 0
  psi := dsolve({init1, init2, schr}, numeric, range = 0 .. 10)
  psiplot := odeplot(psi)
  display(psiplot)
```

Attach a printout of your code with results. You should find a wavefunction with no nodes that blows up at large ξ .

- (c) By decreasing your chosen value of κ^2 to 1.0, you should be able to get the “tail” of the wavefunction to flip over. Since the correct wavefunction should go to zero at large ξ , this means you have bracketed the correct eigenvalue for κ^2 . Choose successively closer together values of κ^2 to find the eigenvalue down to three decimal places. What’s the ground state energy? Attach a plot of your approximate ground state wavefunction.

- (d) Now find the first excited state, which should have exactly one node at $x = 0$. In other words, take initial conditions $\psi(0) = 0$, $\psi'(0) = 1$ (ignoring normalization). Then for $x > 0$, $\psi(x) > 0$ (going to zero at $x \rightarrow \infty$). Use the procedure above starting with $\kappa^2 = 1$ to find the first excited state energy and attach a plot of the wavefunction.

3. Scattering and the Probability Current

We define the probability current $\vec{j}(\vec{x}, t)$ as

$$\vec{j}(x, t) = \frac{i\hbar}{2m} \left(\Psi \vec{\nabla} \Psi^* - \Psi^* \vec{\nabla} \Psi \right) \quad (4)$$

(in 1D, replace the gradient with $\partial\Psi/\partial x$).

- (a) A conserved quantity Q (this could be electric charge or total probability in quantum mechanics) with a density ρ and current \vec{j} satisfies the *continuity equation*

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{j}. \quad (5)$$

Consider an infinite square well potential ($V(x) = 0$ for $-a < x < a$, $V(x) = \infty$ for $|x| > a$). At time $t = 0$, the system is in state $|\Psi(0)\rangle = (|1\rangle + |2\rangle)/\sqrt{2}$, where $|1\rangle$ is the ground state and $|2\rangle$ is the first excited state. Find the $\Psi(x, t)$, probability density $\rho(x, t)$, and probability current $j(x, t)$, and verify that they obey the continuity equation.

- (b) To understand reflection and transmission coefficients when the potential takes different values on either side of the barrier, we should think about conservation of probability. Specifically, write the wavefunction as $Ae^{ikx} + Be^{-ikx}$ to the far left and $Ce^{ik'x}$ to the far right. Show first that the probability current on the far left splits into an incident part ($j_{inc} \propto |A|^2$) and a reflected part ($j_{ref} \propto |B|^2$) and evaluate j_{inc}, j_{ref} . Then find the transmitted probability current j_{trans} . Finally, find the reflection and transmission coefficients $R = j_{ref}/j_{inc}, T = j_{trans}/j_{inc}$ in terms of A, B, C, k, k' .
- (c) *from Griffiths 2.35* Consider a particle moving in a 1D potential

$$V(x) = \begin{cases} 0 & x < 0 \\ -V_0 & x \geq 0 \end{cases}. \quad (6)$$

Determine the reflection and transmission coefficients for a particle incoming from the left (negative x) with energy $E = V_0/3$. Use $R = j_{ref}/j_{inc}, T = j_{trans}/j_{inc}$ and verify that $R + T = 1$