PHYS-4601 Homework 6 Due 22 Oct 2015

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Proofs About Stationary States**

   (a) *Rephrasing Griffiths 2.1(c)* Consider the spatial part of a stationary state $\psi(x)$ (that is, $\Psi(x,t) = \psi(x)e^{-iEt/\hbar}$) and suppose that the potential is an even function of x (ie, $V(x) = V(-x)$). Show that $\psi(\vec{x})$ can be chosen to be either an even or odd function of x. *Hint:* argue that, for any $\psi(x)$ that solves the time-independent Schrödinger equation, so does $\psi(-x)$. Use that to show that the even and odd parts $\psi_\pm(x) = [\psi(x) \pm \psi(-x)]/2$ are also solutions with the same energy.

   (b) *Griffiths 2.2 rephrased* Suppose that the energy E of a stationary state in one dimension is less than the minimum value of the potential. Use the time-independent Schrödinger equation to show that the second derivative of the wavefunction always has the same sign as the wavefunction. Then use that fact to argue qualitatively that such a wavefunction cannot be normalized, proving by contradiction that E must be greater than the minimum value of the potential.

2. **Dirac & the Wall**

   A particle moves in 1D in a potential

$$V(x) = \begin{cases} \infty & x < 0 \\ -\alpha\delta(x-d) & x > 0 \end{cases} \quad \text{with } \alpha > 0 \ . \tag{1}$$

   (a) Assuming that a bound state exists, show that the bound state energy is determined by a transcendental equation

$$\left(\frac{\hbar^2}{m\alpha d}\right) z = 1 - e^{-2z} \ , \tag{2}$$

   where $z = (d/\hbar)\sqrt{-2mE}$.

   (b) Using (2), find the condition that a bound state exists.

   (c) Show that the bound state energy is approximately

$$E = -\frac{m\alpha^2}{2\hbar^2} + \frac{m\alpha^2}{\hbar^2} e^{-2m\alpha d/\hbar^2} \tag{3}$$

   for large d (when the wall is far from the delta function) by solving (2) iteratively assuming that z is large.

3. **Transfer Matrix** *based on Griffiths 2.53*

   Consider a potential in one dimension with $V \neq 0$ in some narrow region centered at $x = 0$ and $V = 0$ both to the left (region 1) and right (region 2). In regions 1 and 2, we can write the wavefunction of a scattering state as

$$\psi_1(x) = A_1 e^{ikx} + B_1 e^{-ikx} \ , \quad \psi_2(x) = A_2 e^{ikx} + B_2 e^{-ikx} \ , \quad k = \sqrt{2mE}/\hbar \ . \tag{4}$$

The coefficients A_i correspond to right-moving waves, and B_i correspond to left-moving waves. If you solve the Schrödinger equation, you will find a linear relationship among the coefficients:

$$\begin{bmatrix} A_2 \\ B_2 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} , \tag{5}$$

where M is called the *transfer matrix.* M is a property of the nonzero potential.

(a) The reflection and transmission coefficients ($R = |B_1/A_1|^2$ and $T = |A_2/A_1|^2$) are found by solving (5) for the case that there are no left-moving waves in region 2 (that is, $B_2 = 0$). We can also define reverse reflection and transmission coefficients $R' = |A_2/B_2|^2$ and $T' = |B_1/B_2|^2$ evaluated when $A_1 = 0$, which are appropriate for waves incoming from the right. Find R, T, R', T' in terms of the transmission matrix elements.

(b) Consdider a delta function well (or barrier) $V(x) = -\alpha\delta(x)$ for real coefficient α of either sign. We know from the class notes that, for normal scattering ($B_2 = 0$),

$$B_1 = \frac{i\beta}{1 - i\beta} A_1 \ , \quad A_2 = \frac{1}{1 - i\beta} A_1 \ , \quad \beta \equiv \frac{m\alpha}{\hbar^2 k} \ . \tag{6}$$

Find the transfer matrix M for a delta function well/barrier. *Hint:* A delta function is an even function, so reverse scattering is physically identical to normal scattering with the role of each coefficient permuted around.

(c) Suppose the nonzero potential consists of two narrow barriers separated by another region of length d with $V = 0$. Show that the overall transfer matrix is $M = U^\dagger M_2 U M_1$, where M_1 and M_2 are the transfer matrices of the two barriers *if they were centered at the origin* and

$$U = \begin{bmatrix} e^{ikd} & 0 \\ 0 & e^{-ikd} \end{bmatrix} . \tag{7}$$

This shows how to calculate reflection and transmission coefficients for a series of barriers. *Hint:* Consider the second barrier is centered at the origin of a redefined coordinate.