PHYS-4601 Homework 5 Due 15 Oct 2015

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Functions of Operators and Time Evolution**

   (a) Suppose $|\lambda\rangle$ is an eigenfunction of $\mathcal{O}$, $\mathcal{O}|\lambda\rangle = \lambda|\lambda\rangle$. For any function $f(x)$ that can be written as a power series
   $$f(x) = \sum_n f_n x^n \ , \tag{1}$$
   we can define
   $$f(\mathcal{O}) = \sum_n f_n \mathcal{O}^n \ . \tag{2}$$
   Show that
   $$f(\mathcal{O})|\lambda\rangle = f(\lambda)|\lambda\rangle \ . \tag{3}$$
   Does this result hold if the power series includes negative powers? Note that this shows that $e^{-iHt/\hbar}|E_n\rangle = e^{-iE_n t/\hbar}|E_n\rangle$ for energy eigenstates $|E_n\rangle$ as claimed in class.

   (b) Show that $U = \exp[iA]$ is unitary, meaning $U^\dagger U = UU^\dagger = 1$, if the operator A is Hermitian (define the exponential by its power series). This shows that the time evolution operator $\exp[-iHt/\hbar]$ is unitary. *Hint:* You may want to show that $(AB)^\dagger = B^\dagger A^\dagger$ for any two operators.

   (c) If we transform our Hilbert space so that $|\Psi'\rangle = U|\Psi\rangle$ for all states $|\Psi\rangle$ and unitary U, show that $\langle\Phi'|\Psi'\rangle = \langle\Phi|\Psi\rangle$. This proves that time evolution via the Schrödinger equation conserves total probability, as it should. We say that quantum mechanics has *unitary time evolution.*

2. **A Two-State System**

   Consider some physical system which only has two states, so its states $|\Psi\rangle$ can be represented by column vectors with two elements. In some basis, the Hamiltonian can be written as
   $$H \simeq \hbar\omega \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \ . \tag{4}$$

   (a) Show that the time evolution operator can be written in this basis as
   $$e^{-iHt/\hbar} \simeq \left(\cos(\omega t) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - i \sin(\omega t) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \ . \tag{5}$$

   (b) Find the stationary states and corresponding energies.

   (c) Consider a general initial state $|\Psi(0)\rangle \simeq \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. First, write this initial state in terms of the stationary states and use that superposition to find the state at a later time t. Then use your result from part (a) to calculate $|\Psi(t)\rangle = e^{-iHt/\hbar}|\Psi(0)\rangle$. Verify that the two results are the same.

3. **Gaussian Wavepacket Part II** *based on Griffiths 2.22*

Here we return to the Gaussian wavepacket in 1D, here looking at the time evolution for the free particle Hamiltonian. We recall from the last assignment that the wavefunction (at some initial time) can be written as

$$|\Psi(t=0)\rangle = \int_{-\infty}^{\infty} dx \, \left(\frac{2a}{\pi}\right)^{1/4} e^{-ax^2}|x\rangle = \int_{-\infty}^{\infty} dp \, \left(\frac{1}{2\pi a\hbar^2}\right)^{1/4} e^{-p^2/4a\hbar^2}|p\rangle \ . \tag{6}$$

(a) Evolve this state in time. First write $\langle p|\Psi(t)\rangle$ and then show that

$$\langle x|\Psi(t)\rangle = \frac{(2a/\pi)^{1/4}}{\sqrt{1+2i\hbar at/m}} e^{-ax^2/(1+2i\hbar at/m)} \ . \tag{7}$$

Hint: You may want to use the trick of "completing the squares" to evaluate a Gaussian integral somewhere.

(b) Find the probability density $|\langle x|\Psi(t)\rangle|^2$. Using the result from assignment 4 that $\langle x^2\rangle = 1/4a$ at $t=0$, find $\langle x^2\rangle$ at a later time t by inspection of the probability density. Qualitatively explain what's happening to the wavefunction as time passes.

(c) What's the momentum-space probability density $|\langle p|\Psi(t)\rangle|^2$? Does $\langle p^2\rangle$ change in time? Does this state continue to saturate the Heisenberg uncertainty relation for $t \neq 0$?