PHYS-4601 Homework 4 Due 8 Oct 2015

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Gaussian Wavepacket Part I**

   Here we take a first look at the Gaussian wavepacket in 1D, which is an important state in more than one physical system. In this problem, we will consider the state at a single instant $t = 0$, ignoring its time evolution. The state is

   $$|\psi\rangle = \int_{-\infty}^{\infty} dx\ Ae^{-ax^2}|x\rangle\ . \tag{1}$$

   (a) Find the normalization constant A. *Hint:* To integrate a Gaussian, consider its square. When you square it, change the dummy integration variable to y, then change the integral over $dxdy$ to plane polar coordinates.

   (b) Since the wavefunction is even, $\langle x\rangle = 0$. Find $\langle x^2\rangle$. *Hint:* You can get a factor of x^2 next to the Gaussian by differentiating it with respect to the parameter a.

   (c) Write $|\psi\rangle$ in the momentum basis. *Hint:* If you have a quantity $ax^2 + bx$ somewhere, you may find it useful to write it as $a(x + b/2a)^2 - b^2/4a$ by completing the square. Then shift integration variables so it looks like you have a Gaussian again.

   (d) Find $\langle p\rangle$ and $\langle p^2\rangle$ and show that this state saturates the Heisenberg uncertainty principle.

2. **The Virial Theorem** *Based on Griffiths 3.31*

   Consider 3D quantum mechanics.

   (a) Using Ehrenfest's theorem, show that

   $$\frac{d}{dt}\langle \vec{x}\cdot\vec{p}\rangle = \left\langle \frac{\vec{p}^2}{m}\right\rangle - \left\langle \vec{x}\cdot\vec{\nabla}V(\vec{x})\right\rangle\ . \tag{2}$$

   (b) Show that the left-hand side of (2) vanishes in a stationary state to prove the *virial theorm*

   $$2\langle K\rangle = \left\langle \vec{x}\cdot\vec{\nabla}V(\vec{x})\right\rangle\ , \tag{3}$$

   where K is the kinetic energy. (The virial theorem holds classically, also, though without the expectation values.)

   (c) Using the virial theorem, find the ratio of (the expectation value of) the kinetic energy to the potential energy for the harmonic oscillator potential $V \propto \vec{x}^2$ and for the Coulomb potential $V \propto 1/|\vec{x}|$. *Hint:* Remember from electromagnetism (or Newton's law of gravity) that $\vec{\nabla}(1/|\vec{x}|) = -\vec{x}/|\vec{x}|^3$.

3. **Measurement vs Time Evolution** *a considerable expansion of Griffiths 3.27*

   Suppose a system has observable A with eigenstates $|a_1\rangle, |a_2\rangle$ of eigenvalues a_1, a_2 respectively and Hamiltonian H with eigenstates $|E_1\rangle, |E_2\rangle$ of energies E_1, E_2 respectively. The eigenstates are related by

   $$|a_1\rangle = \frac{1}{5}\left(3|E_1\rangle + 4|E_2\rangle\right)\ ,\quad |a_2\rangle = \frac{1}{5}\left(4|E_1\rangle - 3|E_2\rangle\right)\ . \tag{4}$$

   Suppose the system is measured to have value a_1 for A initially. Each of the following parts asks about a different possible set of subsequent measurements.

(a) What is the probability of measuring energy E_1 immediately after the first measurement? Assuming we do get E_1, what is the probability of measuring a_1 again if we measure A again immediately after the measurement of energy?

(b) Instead, consider immediately measuring A again after the first measurement. What are the probabilities for observing a_1 and a_2?

(c) Finally, consider making the first measurement and then allowing the system to evolve for time t. If we then measure energy, what is the probability of finding energy E_1? If we instead measured A again, what is the probability we find a_1 again?