PHYS-4601 Homework 21 Due 31 Mar 2016

This homework is due in the dropbox outside 2L26 by 11:59PM on the due date. You may alternately email a PDF (typed or black-and-white scanned) or give a hardcopy to Dr. Frey.

1. **Uniform Gravitational Field** *parts of Griffiths 8.5 and 8.6*

   Consider a ball of mass m that feels a uniform gravitational acceleration g in the $-x$ direction, as by the surface of the earth. Assume that the surface of the earth is at $x = 0$ and forms an infinite potential barrier.

   (a) First, write down what the potential energy is as a function of x.

   (b) Use the WKB approximation to find the allowed energies of the bouncing ball. Find the approximate ground state and first excited state energies in Joules to two significant digits for a neutron (mass $m = 1.7 \times 10^{-27}$ kg). This can actually be measured for ultracold neutrons.

   (c) The *exact* solution of the Schrödinger equation is given by the Airy function

   $$\psi(x) = C\mathrm{Ai}\left[\left(\frac{2m^2 g}{\hbar^2}\right)^{1/3}\left(x - \frac{E}{mg}\right)\right] , \tag{1}$$

   where C is a normalization constant and E is quantized so $\psi(0) = 0$. Denote the zeros of $\mathrm{Ai}(z)$ by a_k ($k = 1, 2, \cdots$ with $|a_1| < |a_2| < \cdots$) and find the energy eigenvalues in terms of the a_k. What are the ground and first excited state energies for a neutron? You will need to look up values of a_k at the Digital Library of Mathematical Functions (DLMF) at `http://dlmf.nist.gov/9.9`.

   (d) Show that the energy eigenvalues match the WKB result in the limit of large quantum number. *Hint:* You can use the asymptotic form of the Airy function itself (either in Griffiths or in the DLMF) or that of the zeros (from the DLMF).

2. **Ionizing an Atom** *from Griffiths 8.16*

   Imagine a hydrogen atom in a small electric field; the electron feels a linear potential from the field, which eventually becomes less than the ground state energy, so it can tunnel out of the atom. In this problem, consider a simple 1D model of this system, with potential

   $$V(x) = \begin{cases} \infty\ , & x < -a \\ -V_0\ , & -a < x < 0 \\ -\alpha x\ , & x > 0 \end{cases} . \tag{2}$$

   (a) Suppose the square well is very deep, so $V_0 \gg \hbar^2/ma^2$. In the absence of the electric field ($\alpha = 0$), what is the approximate ground state energy E? If the electron were a classical particle with this kinetic energy, what would be its speed? *Hint:* You can think of this as the energy of the first odd eigenfunction of a finite square well of width $2a$ or you can approximate the potential as nearly an infinite square well.

   (b) Show that the lifetime of the atom in the presence of the field is $\ln\tau = A|E|^{3/2} + B$, where A and B are constants. Then find A and B (you may need your results from part (a)).

3. **WKB Bloch Waves**

   Consider an electron moving in a 1D potential $V(x) = V_0 \sin^2(\pi x/a)$.

(a) If the particle has energy $E > V_0$, find the right-moving wavefunction in the WKB approximation in terms of the *incomplete elliptic integral of the second kind*

$$\hat{E}(\phi, \beta) \equiv \int_0^\phi d\theta \sqrt{1 - \beta^2 \sin^2\theta} \quad (0 < \beta \leq 1) \ . \tag{3}$$

Do not normalize the wavefunction. (Usually the elliptic integral is denoted E, but we add a hat to distinguish it from energy.)

(b) Ignoring normalization, use Maple to plot the real part of your wavefunction as a function of x/a from 0 to 10 assuming $2mV_0a^2/\hbar^2 = 9$ in the two cases that $E = 2V_0$ and $E = 10V_0$. Attach a print out of your Maple code. *Note:* Be careful in your code; the Maple command `EllipticE` takes arguments $\sin\phi$ and β, not ϕ and β.

(c) Use angle addition formulas to prove that

$$\hat{E}(\phi + n\pi, \beta) = \hat{E}(\phi, \beta) + 2n\hat{E}(\pi/2, \beta) \tag{4}$$

for n an integer. Then use this to show that your wavefunction satisfies Bloch's theorem for a periodic potential (see our discussion of solid state physics). *Hint:* Break up the integral and use the fact that $\sin\theta$ is symmetric around $\theta = \pi/2$. Also note that the periodicity of the potential is a.